

MOLLIFIER SMOOTHING OF TENSOR FIELDS ON DIFFERENTIABLE MANIFOLDS AND APPLICATIONS TO RIEMANNIAN GEOMETRY

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Dedicated to Professor Francesco Mercuri on his sixtieth birthday

ABSTRACT. Let M be a differentiable manifold. We say that a tensor field g defined on M is non-regular if g is in some local L^p space or if g is continuous. In this work we define a mollifier smoothing g_ε of g that has the following feature: If g is a Riemannian metric of class C^2 , then the Levi-Civita connection and the Riemannian curvature tensor of g_ε converges to the Levi-Civita connection and to the Riemannian curvature tensor of g respectively as ε converges to zero. Therefore this mollifier smoothing is a good starting point in order to generalize objects of the classical Riemannian geometry to non-regular Riemannian manifolds. Finally we give some applications of this mollifier smoothing. In particular, we generalize the concept of Lipschitz-Killing curvature measure for some non-regular Riemannian manifolds.

1. INTRODUCTION

The concept of Riemannian manifold can be generalized in several ways. Finsler manifolds, sets with positive reach in Euclidean spaces and Alexandrov spaces are examples of generalizations of some classes of Riemannian manifolds. A good starting point for the reader who is interested in this subject is M. Berger's book [2] and references therein. In this work we introduce the L^p_{loc} tensor fields and the continuous tensor fields on differentiable manifolds, which we call *non-regular tensor fields*. We are particularly interested to study non-regular Riemannian metrics \hat{g} on n -dimensional differentiable manifolds M^n . We call a pair (M^n, \hat{g}) by *non-regular Riemannian manifold*.

As an example of an L^p_{loc} Riemannian manifold (that is, a differentiable manifold with an L^p_{loc} Riemannian metric), fix a convex set $D \subset \mathbb{R}^n$ and suppose that it is closed and bounded. Its boundary ∂D is called a convex surface. It is well known that ∂D have a tangent space almost everywhere and we can define an inner product from the ambient space on each tangent space. It is not difficult to see that ∂D endowed with this field of inner products is a non-regular Riemannian manifold. Eqs. (29), (30) and (31) give other examples of L^p_{loc} Riemannian manifolds.

Non-regular Riemannian manifolds can not be studied through differentiation as in classical Riemannian geometry. However the space of non-regular tensor fields can be topologized in the same way as the space of non-regular functions. The main idea of this work is to study non-regular Riemannian manifolds through approximations by (smooth) Riemannian manifolds. Let (M, \hat{g}) be a non-regular Riemannian manifold and suppose that $\{(M, \hat{g}(\varepsilon)), \varepsilon > 0\}$ is a one-parameter family of Riemannian manifolds such that

2000 *Mathematics Subject Classification.* Primary 53B21; Secondary 41A35, 53A45, 53B20.

Key words and phrases. Mollifier smoothing, Non-regular Riemannian metrics, Levi-Civita connection, Riemannian curvature tensor, Lipschitz-Killing curvature measures.

$\widehat{g}(\varepsilon)$ converges to \widehat{g} in $L^p_{\text{loc}}(M)$ (or in $C^0_{\text{loc}}(M)$) as ε goes to zero. We can try to define some geometric object in (M, \widehat{g}) studying the behavior of the correspondent object in the family $\{(M, \widehat{g}(\varepsilon)), \varepsilon > 0\}$ as ε goes to zero. As an illustration of this approach we can imagine the round sphere of radius 1 in \mathbb{R}^3 deforming gradually and converging to a cube. Observe that the total curvature during the process is equal to 4π . Moreover the curvature concentrates around the vertices of the cube while the deformation follows, because the other points are becoming flat. If the deformation is made “symmetrically”, then the total curvature in a neighborhood of each vertex is equal to $\pi/2$. Therefore the Gaussian curvature of a vertex p can be thought as $(\pi/2)$. In this way, we “were able to generalize” the concept of Gaussian curvature for the cube.

The distance between two points in a non-regular Riemannian manifold (M, \widehat{g}) is also defined using approximation by (smooth) Riemannian manifolds (See Definition 6.4 and Remark 6.5). If we identify the points with zero distance, then the resulting identification space is a metric space that is not necessarily homeomorphic to a differentiable manifold. We can see Eqs. (29) and (31) again to have an idea of which kind of metric spaces a non-regular Riemannian manifold can generate.

Let us explain how this work is organized. Meanwhile, we cite some related works. In Section 2, we fix some notations and present some classical results that will be used afterwards. In Section 3, we establish the foundations of non-regular tensor fields on differentiable manifolds. In Section 4, we introduce the mollifier smoothing of a non-regular tensor field \widehat{T} in a Riemannian manifold (M, \widehat{g}) . A smooth Riemannian metric \widetilde{g} , which is called the *background metric*, is introduced as an “auxiliary” metric which is necessary to define the mollifier smoothing. We are particularly interested when \widehat{T} is a non-regular Riemannian metric.

In Section 5 we adapt the mollifier smoothing introduced in Section 4 in the following way: Take a locally finite covering of M by open sets with Euclidean background metric. Besides, take a partition of the unity subordinated to this covering. We call this covering with this partition of the unity by \mathcal{P} . Consider the mollifier smoothing defined in Section 4 on each open set of the covering. The sum of these mollifier smoothings weighted by the partition of the unity is called *mollifier smoothing with respect to \mathcal{P}* . It has the following interesting property: If the non-regular Riemannian metric \widehat{g} is of class C^2 , then the Levi-Civita connection and the Riemannian curvature tensor of the mollifier smoothing \widehat{g}_ε with respect to \mathcal{P} converges to the Levi-Civita connection and the Riemannian curvature tensor of \widehat{g} respectively as ε goes to zero. Therefore this mollifier smoothing provides a natural starting point in order to generalize objects of the classical Riemannian Geometry to non-regular Riemannian manifolds.

It is interesting to present other mollifier smoothings that we can find in the literature: The mollifier smoothing of distributions (See the regularization “ R ” in [8]) is quite similar to the mollifier smoothing with respect to \mathcal{P} . The main difference is that the “gluing” of local smoothings is done there by composition of the local smoothings. Another work where we find a mollifier smoothing is in Nash’s celebrated work about embeddings of Riemannian manifolds into Euclidean spaces (see [14]). There he defines a smoothing of tensor fields on Riemannian manifolds. He embeds the manifold into an Euclidean space and he makes the convolution of the tensor field with a mollifier. This mollifier is defined on the ambient space and it decays quickly as it goes to infinity. This convolution eliminates the “high frequencies” of the original tensor field, what is typical in Fourier

analysis. Our definition of mollifier smoothing is intrinsic and our kernel has compact support. Finally another work that is worth mentioning (although it is not directly related with our work) is Karcher's paper [11]. There the author introduces a mollifier smoothing of a map between Riemannian manifolds using the center of mass.

The last sections are devoted to some applications. We do not want to be extensive there. Our aim is to convince the reader that the mollifier smoothings of tensor fields given in Definitions 4.3 and 5.1 can be useful in order to study non-regular Riemannian metrics and its singularities. The author hopes that this theory can provide useful examples to the people who study the convergence of sequences of Riemannian metrics, although the author's lack of knowledge in this field does not allow him to say anything deeper about this subject.

In Section 6 we generalize the concept of distance between two points for non-regular Riemannian manifolds. There are other works that study the concept of distance for non-regular manifolds. For instance in [3] and [4], Cecco and Palmieri generalize the concept of distance on Lipschitz manifolds with a Riemannian metric. There the non-regularity is imposed by a Lipschitz atlas, which is an atlas with Lipschitz change of coordinates. Particularly interesting is the definition of distance between two points given in [4], which depends on an integral over (M, \hat{g}) .

In Section 7 we generalize the parallel transport for some non-regular Riemannian manifolds using mollifier smoothing with respect to \mathcal{P} . We prove that it is well defined for some curves on piecewise smooth two-dimensional Riemannian manifolds (see Definition 7.6). For the reader who is interested in other works about parallel transport in less regular spaces, Nikolaev and Petrunin study the parallel transport in Alexandrov spaces in [15] and [16].

In Section 8 we define the Lipschitz-Killing curvature measures for closed non-regular Riemannian manifolds (M, \hat{g}) (see Definition 8.11). It is a measure that represents the total Lipschitz-Killing curvature on Borel sets in M . The existence of the Lipschitz-Killing curvature measures depends on the existence of a Lipschitz-Killing curvature measure generator, which is essentially a family of "regular" open sets of M that generates the topology of M (See Definition 8.1 for more details). The Lipschitz-Killing curvature measure for these generators is defined using the mollifier smoothing with respect to \mathcal{P} . Then we extend the Lipschitz-Killing curvature measure to Borel subsets of M .

In [5], Cheeger, Müller and Schrader study sequences of piecewise flat manifolds that converges to a smooth Riemannian manifold (M, g) . Although this seems to be the opposite we are doing here, the general idea is the same: to approximate a Riemannian manifold by simpler objects. There the authors prove that the Lipschitz-Killing curvature of some sequences of piecewise flat manifolds converges, in the sense of measures, to the Lipschitz-Killing curvatures of the smooth Riemannian manifold. A similar convergence holds for the mean curvatures of the boundary.

The concept of curvature measures is also used, for instance, in the Federer's theory of subsets with positive reach in \mathbb{R}^n (See [10]). Let $A \subset \mathbb{R}^n$ be a subset with positive reach and let $B \subset A$ be a Borel subset of \mathbb{R}^n . The coefficients of the Steiner polynomial of $B \subset A$ are essentially the curvature measures of $B \subset A$ and some of these coefficients are essentially generalizations of the Lipschitz-Killing curvature measures for the case of smooth hypersurfaces.

Another instance where the concept of curvature measure appear is in Alexandrov theory of convex surfaces (See [1]). There the Gaussian curvature measure is defined in

the following way: in open geodesic triangles it is defined as 2π minus the sum of the internal angles; in open geodesic arcs it is defined as zero; at the points it is defined as 2π minus the total angle at the point. From these curvature measures, we are able to extend the total Gaussian curvature to Borel subsets of the surface.

In Section 9 we study piecewise smooth two-dimensional Riemannian manifolds more deeply. They are essentially polyhedras with non-flat faces and edges. We get a Gaussian curvature measure generator for these kind of surfaces and we prove that the Gaussian curvature measure has the “expected” geometrical values. Meanwhile we get an alternative proof of a (well known) generalization of the Gauss-Bonnet theorem for these kind of surfaces (See Theorem 9.3).

It is completely natural to conjecture that the Lipschitz-Killing curvature measures we define here coincides with the classical definitions if the geometrical object lies in the intersection of both theories.

In Section 10, we present an odd example of a two-dimensional sphere with a Riemannian metric of class C^1 such that it is flat outside a subset with Hausdorff dimension $(1 + (\ln 2 / \ln 3))$. In other words, its curvature is “concentrated” in a subset with Hausdorff dimension $(1 + (\ln 2 / \ln 3))$, in the same way that the curvature of a cube is concentrated on its vertices (zero dimensional subsets). This sphere shows a strange behaviour that a curvature can assume in non-regular Riemannian manifolds. It can also be useful in order to study the “dimensional character” of the curvature at a point, like the zero-dimensional curvature at the vertices, the one dimensional curvature at the edges and the two-dimensional curvature at the faces of a piecewise smooth two-dimensional Riemannian manifold.

2. PRELIMINARIES

In this section we fix some notations and remember some classical theorems that will be used afterwards. The material exposed here can be found in any good textbook of Riemannian Geometry and Measure Theory, for instance [6], [7], [9], [12], [17], [18] and [20]. The exception is Theorem 2.1, which proof is given here. In Riemannian geometry, there exist two definitions of curvature tensor which differ by a sign. We adopt the convention given in [12].

2.1. Riemannian geometry. Let M^n be an n -dimensional differentiable manifold (The superscript n is omitted whenever there is not any possibility of misunderstandings). Denote the tangent bundle over M by TM , the cotangent bundle over M by T^*M and the tensor bundle of type (m, s) over M by $T^{m,s}M$. The respective fibers over $x \in M$ is denoted by T_xM , T_x^*M and $T_x^{m,s}M$. If $\varphi_1, \dots, \varphi_m \in TM$, $v_1, \dots, v_s \in T^*M$ and $T \in T^{m,s}M$, then $T(\varphi_1, \dots, \varphi_m, v_1, \dots, v_s)$ is the contraction of these tensor fields.

Let (M, g) be an n -dimensional Riemannian manifold with smooth metric g . As usual we use the notation $\langle u, v \rangle_g := g(u, v)$ for the scalar product of the vectors u and v and the subscript g will be omitted whenever there is not any possibility of misunderstandings. The symbol ∇ denote the Levi-Civita connection and R denote the curvature tensor, where $R(u, v) = \nabla_u \nabla_v - \nabla_v \nabla_u - \nabla_{[u, v]}$. The Ricci tensor is defined by $Ric(u, v) = \sum_{i=1}^n \langle R(w_i, u)v, w_i \rangle$ and the scalar curvature by $S = \sum_{i,j=1}^n \langle R(w_i, w_j)w_j, w_i \rangle$, where $\{w_1, \dots, w_n\}$ is an orthonormal basis of T_xM .

Put a coordinate system (x_1, \dots, x_n) in a neighborhood of $x \in M$. The coordinate vector fields is denoted by $\partial/\partial x_i, i = 1, \dots, n$. The components of the metric with respect to this coordinate system is denoted by $g_{ij} = g(\partial/\partial x_i, \partial/\partial x_j)$, $i, j = 1, \dots, n$. The Christoffel symbols are denoted by Γ_{ij}^k and they are defined implicitly as

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^n \Gamma_{ij}^k \frac{\partial}{\partial x_k}.$$

It is well known that

$$(1) \quad \Gamma_{ij}^k = \frac{1}{2} \sum_{m=1}^n \left\{ \frac{\partial}{\partial x_i} g_{jm} + \frac{\partial}{\partial x_j} g_{mi} - \frac{\partial}{\partial x_m} g_{ij} \right\} g^{km}$$

where g^{km} are the components of the inverse matrix of g .

The components of the curvature tensor are given implicitly by

$$(2) \quad R \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) \frac{\partial}{\partial x_k} = \sum_{l=1}^n R_{kij}^l \frac{\partial}{\partial x_l}.$$

It is well known that

$$(3) \quad R_{kij}^l = \left(\frac{\partial \Gamma_{jk}^l}{\partial x_i} - \frac{\partial \Gamma_{ik}^l}{\partial x_j} \right) + \sum_{m=1}^n (\Gamma_{jk}^m \Gamma_{im}^l - \Gamma_{ik}^m \Gamma_{jm}^l).$$

Let $\{w_1, \dots, w_n\}$ be an orthonormal moving frame for the tangent bundle of (M, g) and denote the dual frame by $\{\varpi_1, \dots, \varpi_n\}$. Define the curvature forms Ω_{lk} for the orthonormal moving frame $\{w_1, \dots, w_n\}$ by

$$(4) \quad \sum_{l=1}^n \Omega_{lk}(w_i, w_j) w_l = R(w_i, w_j) w_k.$$

For $\kappa = 1, \dots, n$, the κ -th Lipschitz-Killing curvature measure \mathcal{R}^κ is a measure defined on open subsets $U \subset M$: if κ is odd, then \mathcal{R}^κ is identically zero; if κ is even then

$$(5) \quad \mathcal{R}^\kappa(U) = \int_U R^\kappa$$

where

$$(6) \quad R^\kappa = \frac{(-1)^{\frac{\kappa}{2}}}{(n - \kappa)! 2^\kappa \pi^{\frac{\kappa}{2}} (\kappa/2)!} \sum_{\sigma} (-1)^{|\sigma|} \Omega_{\sigma(1)\sigma(2)} \wedge \dots \wedge \Omega_{\sigma(\kappa-1)\sigma(\kappa)} \wedge \varpi_{\sigma(\kappa+1)} \wedge \dots \wedge \varpi_{\sigma(n)},$$

and the summation is over all permutations of n elements. Notice that R^κ does not depend on the orthonormal moving frame we choose. The curvature measure \mathcal{R}^0 is the volume and \mathcal{R}^2 is proportional to the total scalar curvature.

The volume element of (M, g) is denoted by $dV_g(\cdot)$. When (M, g) is the Euclidean space with its canonical metric, the volume element is simply denoted by $dV(\cdot)$. $B_g(x, r) := \{y \in M; \text{dist}_g(x, y) < r\}$ denote the geodesic ball with center x and radius r .

The exponential map is denoted by $\exp : \Psi M \subset TM \rightarrow M$. The exponential map restricted to the tangent space $T_x M$ is denoted by $\exp_x : \Psi_x M \subset T_x M \rightarrow M$.

Suppose that there exist a unique minimizing geodesic γ connecting x and y in (M, g) . We denote the parallel transport between the tensor spaces $T_x^{m,s} M$ and $T_y^{m,s} M$ through γ by $\tau_{x,y}$. The following theorem states essentially that the parallel transport inside the

injectivity radius is smooth with respect to all its parameters. Although natural, it is not usually found as stated here.

THEOREM 2.1. *Let (M^n, g) be an n -dimensional Riemannian manifold with smooth Riemannian metric g . Let q_1 and q_2 be points in M and γ be the unique minimizing geodesic joining them such that $\exp_{q_1}(\gamma'(0)) = q_2$. Then there exists a neighborhood N_{q_1} of $q_1 \in M$ and N_{q_2} of $q_2 \in M$ such that the map $\tau : TN_{q_1} \times N_{q_2} \rightarrow M \times TM$ defined by $\tau((x, v), y) = (x, (y, \tau_{x,y}(v)))$ is smooth.*

Proof. Set $TMM = \{(x, \dot{x}, v); x \in M, (\dot{x}, v) \in T_x M \times T_x M\}$. Define $\rho_1 : U \subset TMM \rightarrow M \times TM$ which is given by $\rho_1(x, \dot{x}, v) = (x, (\exp(x, \dot{x}), \tau_{x, \exp(x, \dot{x})} v))$, where U is chosen such that ρ_1 is well defined.

Parametrize U by $(x_1, x_2, \dots, x_n, \dot{x}_1, \dots, \dot{x}_n, v_1, \dots, v_n)$. The function ρ_1 is smooth because $\rho_1(x_0, \dot{x}_0, v_0)$ is the solution of the system of ordinary differential equations

$$\begin{aligned} \frac{d\dot{x}_k}{dt} + \sum_{i,j} \Gamma_{ij}^k \dot{x}_i \dot{x}_j &= 0 & k = 1, \dots, n \\ \frac{dx_d}{dt} &= \dot{x}_d & d = 1, \dots, n \\ \frac{dv_l}{dt} + \sum_{i,j} \Gamma_{ij}^l \dot{x}_i v_j &= 0 & l = 1, \dots, n \end{aligned}$$

with initial conditions $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$ and $v(0) = v_0$, which is smooth with respect to the initial conditions.

Let q_1 , q_2 and γ as in the hypothesis of the theorem. Observe that the Jacobian matrix of ρ_1 at $(q_1, \tilde{\gamma}'(0), v)$ is nonsingular for every $v \in T_{q_1} M$. Therefore there exists a neighborhood U_1 of $(q_1, \tilde{\gamma}'(0), v)$ such that $\rho_1|_{U_1} : U_1 \rightarrow \rho_1(U_1)$ is a diffeomorphism. Observe that we can pick $U_1 = \{(x, \dot{x}, v)\}$ such that every $v \in T_x M$ is included.

Now define $\rho_2 : U \subset TMM \rightarrow TM \times M$ which is given by $\rho_2(x, \dot{x}, v) = ((x, v), \exp(x, \dot{x}))$ where U is chosen such that ρ_2 is well defined. Using the same argument as before, if we take q_1 , q_2 and γ as in the hypothesis of the theorem, then there exists a neighborhood U_2 of $(q_1, \gamma'(0), v)$ such that $\rho_2|_{U_2} : U_2 \rightarrow \rho_2(U_2)$ is a diffeomorphism. Observe that we can also pick $U_2 = \{(x, \dot{x}, v)\}$ such that every $v \in T_x M$ is included.

Finally observe that there exist neighborhoods N_{q_1} of $q_1 \in M$ and N_{q_2} of $q_2 \in M$ such that $\tau := \rho_1 \circ \rho_2^{-1}$ defined on $TN_{q_1} \times N_{q_2}$ is smooth. □

2.2. Measure theory.

THEOREM 2.2. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a locally summable function. Then for a.e. $x \in \mathbb{R}^n$, we have that*

$$(7) \quad \frac{\int_{B(x,r)} |f(y) - f(x)| dV(y)}{\text{Vol}(B(x,r))} \rightarrow 0$$

as $r \rightarrow 0$. Such a point x is called a Lebesgue point of f .

DEFINITION 2.3. We say that an open subset $U \subset M$ is compactly contained in M if the closure \bar{U} of U is compact. In this case, we denote $U \subset\subset M$.

REMARK 2.4. The Lebesgue's Differentiation Theorem is valid for smooth Riemannian manifolds (M^n, g) . In fact, take a coordinate system $\phi : U \rightarrow \mathbb{R}^n$ in $U \subset\subset M$ such that $\phi(U) \subset\subset \mathbb{R}^n$. Consider three metrics in U : g_{ij} , $c.\delta_{ij}$ and $C.\delta_{ij}$. Here δ_{ij} is the Euclidean metric with respect to ϕ and the constants c and C are chosen in such a way that $\|v\|_{c\delta} \leq \|v\|_g \leq \|v\|_{C\delta}$ for every $v \in TU$. Suppose that $x \in U$ is a Lebesgue point of f in $(M, c\delta)$. Then

$$\begin{aligned} \frac{\int_{B_g(x,r)} |f(y) - f(x)| \cdot dV_g(y)}{\int_{B_g(x,r)} dV_g(y)} &\leq \frac{\int_{B_g(x,r)} |f(y) - f(x)| \cdot dV_{C\delta}(y)}{\int_{B_g(x,r)} dV_{c\delta}(y)} \\ &\leq \frac{\int_{B_{c\delta}(x,r)} |f(y) - f(x)| \cdot dV_{C\delta}(y)}{\int_{B_{c\delta}(x,r)} dV_{c\delta}(y)} \leq \frac{C^n}{c^n} \frac{\int_{B_{c\delta}(x,r)} |f(y) - f(x)| \cdot dV_{c\delta}(y)}{\int_{B_{c\delta}(x,r)} dV_{c\delta}(y)} \\ &\leq \frac{C^{2n}}{c^{2n}} \frac{\int_{B_{c\delta}(x,r)} |f(y) - f(x)| \cdot dV_{c\delta}(y)}{\int_{B_{c\delta}(x,r)} dV_{c\delta}(y)} \rightarrow 0 \quad \text{as } r \rightarrow 0. \end{aligned}$$

Therefore every Lebesgue point of f in $(U, c\delta)$ is a Lebesgue point of f in (U, g) , what proves the Lebesgue's Differentiation Theorem for the Riemannian case.

We will need the following variation of the Lebesgue differentiation theorem:

PROPOSITION 2.5. *Let (M, g) be a Riemannian manifold and $f : M \rightarrow \mathbb{R}$ be a locally summable function. Consider $x \in M$ a Lebesgue point of f and let $h : M \rightarrow \mathbb{R}$ be a continuous function. Then x is also a Lebesgue point of the product $h.f$.*

Proof.

$$\begin{aligned} &\frac{\int_{B_g(x,r)} |h(y)f(y) - h(x)f(x)| dV_g(y)}{\text{Vol}(B_g(x,r))} \\ &\leq \frac{\int_{B_g(x,r)} |h(y)f(y) - h(y)f(x)| dV_g(y)}{\text{Vol}(B_g(x,r))} \\ &\quad + \frac{\int_{B_g(x,r)} |h(y)f(x) - h(x)f(x)| dV_g(y)}{\text{Vol}(B_g(x,r))} \end{aligned}$$

which obviously goes to zero as r goes to zero. \square

Now we state Fatou's Lemma and Fubini's Theorem according to our necessities.

THEOREM 2.6. (Fatou's Lemma) *Let (M, g) be a Riemannian manifold and consider a sequence of non-negative real valued functions $\{f_i\}_{i \in \mathbb{N}}$. Assume that $\liminf_{i \rightarrow \infty} \|f_i\|_{L^1(M,g)}$ exists. Then $\liminf_{i \rightarrow \infty} f_i(x)$ exists for almost every x , the function $\liminf_{i \rightarrow \infty} f_i$ is in $L^1(M, g)$, and we have that*

$$\int_M (\liminf_{i \rightarrow \infty} f_i)(x) dV_g(x) \leq \liminf_{i \rightarrow \infty} \int_M f_i(x) dV_g(x) \leq \liminf_{i \rightarrow \infty} \|f_i\|_{L^1(M,g)}.$$

THEOREM 2.7. (Fubini's Theorem) *Let (M_1, g_1) and (M_2, g_2) be two smooth Riemannian manifolds and let $f : M_1 \times M_2 \rightarrow \mathbb{R}$ be a Lebesgue summable function with respect to the product Riemannian metric. Then $f(x, \cdot) : M_2 \rightarrow \mathbb{R}$ are summable for almost*

every $x \in M_1$ and $f(\cdot, y) : M_1 \rightarrow \mathbb{R}$ are summable for almost every $y \in M_2$. Moreover $\int_{M_2} f(x, y) dV_{g_2}(y)$ is summable on M_1 , $\int_{M_1} f(x, y) dV_{g_1}(x)$ is summable on M_2 and

$$\begin{aligned} \int_{M_1 \times M_2} f(x, y) dV_{g_1}(x) \cdot dV_{g_2}(y) &= \int_{M_1} \left(\int_{M_2} f(x, y) dV_{g_2}(y) \right) dV_{g_1}(x) \\ &= \int_{M_2} \left(\int_{M_1} f(x, y) dV_{g_1}(x) \right) dV_{g_2}(y). \end{aligned}$$

Finally we remember the definition of a measure on a sigma-algebra.

DEFINITION 2.8. Let M be an differentiable manifold. We say that a collection \mathcal{L} of subsets of M is a σ -algebra in M if

- (1) $M \in \mathcal{L}$.
- (2) $(M - A) \in \mathcal{L}$ whenever $A \in \mathcal{L}$.
- (3) If $\{A_i \in \mathcal{L}\}_{i \in \mathbb{N}}$ is a countable family, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{L}$.

THEOREM 2.9. For every collection \mathcal{M} of subsets of M , there exists the smallest σ -algebra that contains \mathcal{M} .

DEFINITION 2.10. The smallest σ -algebra that contains all open sets of M is denoted by $\mathcal{B}(M)$ and its elements are called *Borel sets* of M .

DEFINITION 2.11. Let \mathcal{L} be a σ -algebra of M . A *positive measure* is a non-negative function $\mathcal{S} : \mathcal{L} \rightarrow [0, \infty]$ which is *countably additive*, that is, if $\{A_i \in \mathcal{L}\}_{i \in \mathbb{N}}$ is a countable family of pairwise disjoint subsets of M , then

$$\mathcal{S}\left(\bigcup_{i=1}^{\infty} A_i, g\right) = \sum_{i=1}^{\infty} \mathcal{S}(A_i, g).$$

Finally we say that $\mathcal{S} : \mathcal{L} \rightarrow [-\infty, \infty]$ is a *signed measure* if it is a difference of two positive measures and if this difference is always well defined (that is, the indetermination $\infty - \infty$ never happens).

3. NON-REGULAR TENSOR FIELD SPACES

In this section, we develop the foundations of non-regular tensor field spaces.

DEFINITION 3.1. We say that a tensor field T of type (m, s) on a differentiable manifold M is continuous if the real valued function $T(\varphi_1, \dots, \varphi_m, v_1, \dots, v_s) : M \rightarrow \mathbb{R}$ is continuous for every family $\{\varphi_i : M \rightarrow T^*M; 1 \leq i \leq m\}$ of smooth 1-forms and every family $\{v_i : M \rightarrow TM; 1 \leq i \leq s\}$ of smooth vector fields. We denote the family of continuous tensor fields by $C_{\text{loc}}^0(M)$.

Let $f : M \rightarrow \mathbb{R}$ be a measurable function. If we put a smooth Riemannian metric g on M , it makes sense to talk about the space of functions such that $|f|^p$ is locally integrable. But the local integrability of $|f|^p$ does not depend on the choice of g . Therefore it makes sense to talk about $L_{\text{loc}}^p(M)$ spaces on a *differentiable* manifold M . The same situation happens with tensor fields.

DEFINITION 3.2. We say that a tensor field T of type (m, s) on a differentiable manifold M is in $L^p_{\text{loc}}(M)$ if the function $T(\varphi_1, \dots, \varphi_m, v_1, \dots, v_s) : M \rightarrow \mathbb{R}$ is in $L^p_{\text{loc}}(M)$ for every family $\{\varphi_i : M \rightarrow T^*M; 1 \leq i \leq m\}$ of smooth 1-forms and every family $\{v_i : M \rightarrow TM; 1 \leq i \leq s\}$ of smooth vector fields. When $T \in L^1_{\text{loc}}(M)$, we say that T is locally summable.

DEFINITION 3.3. We say that a tensor field is *non-regular* if it is in either $L^p_{\text{loc}}(M)$ or $C^0_{\text{loc}}(M)$.

REMARK 3.4. It is easy to see that every $C^0_{\text{loc}}(M)$ tensor field is an $L^p_{\text{loc}}(M)$ tensor field. But we continue to make the distinction between these spaces because they are topologically different.

We intend to define the mollifier smoothing \widehat{T}_ε of a non-regular tensor field \widehat{T} . In order to do it, we put a smooth Riemannian metric \widetilde{g} on M , which we call *the background metric*. Although most of our results does not depend on \widetilde{g} , we have to develop a theory about L^p and C^0 tensor fields on a Riemannian manifold (M, \widetilde{g}) . $\widetilde{\nabla}$ will denote the Levi-Civita connection with respect to \widetilde{g} .

REMARK 3.5. Symbols which comes with “hat”, such as \widehat{g} and \widehat{T} , stand for non-regular tensor fields. The symbol \widetilde{g} stand for the background metric. Symbols without any superscript, such as g and T , stand for general tensor fields.

We denote the “sup” norm in several situations by $\|\cdot\|_{C^0(M, \widetilde{g})}$. When there is not any possibility of misunderstandings, we simplify $\|\cdot\|_{C^0(M, \widetilde{g})}$ by $\|\cdot\|_{C^0}$. The following generalizations of L^p and C^0 spaces for the tensorial case are very natural:

DEFINITION 3.6. We say that the tensor field T of type (m, s) is in $C^0(M, \widetilde{g})$ if the function $T(\varphi_1, \dots, \varphi_m, v_1, \dots, v_s) : M \rightarrow \mathbb{R}$ is continuous and bounded for every family of smooth 1-forms $\{\varphi_j : M \rightarrow T^*M; 1 \leq j \leq m, \|\varphi_j\|_{C^0(M, \widetilde{g})} \leq 1\}$ and every family of smooth vector fields $\{v_i : M \rightarrow TM; 1 \leq i \leq s, \|v_i\|_{C^0(M, \widetilde{g})} \leq 1\}$.

DEFINITION 3.7. Let T be a tensor field of type (m, s) defined on (M, \widetilde{g}) . We say that T is in $L^p(M, \widetilde{g})$ if the function $T(\varphi_1, \dots, \varphi_m, v_1, \dots, v_s) : M \rightarrow \mathbb{R}$ is in $L^p(M, \widetilde{g})$ for every family of smooth 1-forms $\{\varphi_j : M \rightarrow T^*M; 1 \leq j \leq m, \|\varphi_j\|_{C^0(M, \widetilde{g})} \leq 1\}$ and every family of smooth vector fields $\{v_i : M \rightarrow TM; 1 \leq i \leq s, \|v_i\|_{C^0(M, \widetilde{g})} \leq 1\}$. Here we identify the tensor fields that coincides almost everywhere. When $T \in L^1(M, \widetilde{g})$, we say that T is summable.

REMARK 3.8. Let (M, \widetilde{g}) be a differentiable manifold with background metric \widetilde{g} . Observe that:

- (1) T is continuous if and only if $T \in C^0(U, \widetilde{g}|_U)$ for every $U \subset\subset M$.
- (2) $T \in L^p_{\text{loc}}(M)$ if and only if $T \in L^p(U, \widetilde{g}|_U)$ for every $U \subset\subset M$.

The relationships $T \in C^0(U, \widetilde{g}|_U)$ and $T \in L^p(U, \widetilde{g}|_U)$ do not depend on the background metric \widetilde{g} we choose. In particular, if M is a closed differentiable manifold, then the relationships $T \in C^0(M, \widetilde{g})$ and $T \in L^p(M, \widetilde{g})$ do not depend on the background metric \widetilde{g} we choose.

We will introduce complete norms on the tensor field spaces $C^0(M, \widetilde{g})$ and $L^p(M, \widetilde{g})$, which generalize the $C^0(M, \widetilde{g})$ and $L^p(M, \widetilde{g})$ norms for real valued function spaces. The

proof of the completeness of these norms will be given in Theorems 3.12 and 3.13 respectively. Although we are mainly interested to work with the spaces $C_{\text{loc}}^0(M)$ and $L_{\text{loc}}^p(M)$, Theorems 3.12 and 3.13 are interesting by themselves and they are essential for the study of $C_{\text{loc}}^0(M)$ and $L_{\text{loc}}^p(M)$ spaces.

Let us introduce a covering of M by coordinate neighborhoods, which will be useful afterwards. The general idea is that using these coordinate systems, the local problem come out to be essentially Euclidean. Afterwards we can glue each part using a partition of the unity.

DEFINITION 3.9. Let (M^n, \tilde{g}) be a Riemannian manifold, $U \subset M$ an open set and $\phi : U \rightarrow \mathbb{R}^n$ a coordinate system. If v is a smooth vector field and φ is a smooth 1-form defined on M , then their restrictions to U are written as

$$(8) \quad \varphi|_U = \sum_{i=1}^n a_i(x, \varphi) \cdot dx_i(x)$$

and

$$(9) \quad v|_U = \sum_{j=1}^n b_j(x, v) \cdot \frac{\partial}{\partial x_j}(x)$$

respectively. We say that ϕ is an almost Euclidean coordinate system if

- (1) $\frac{1}{2} \leq \|dx_i(x)\|_{C^0(M, \tilde{g})} \leq 2$ for every $i = 1 \dots n$.
- (2) $\frac{1}{2} \leq \|\frac{\partial}{\partial x_j}(x)\|_{C^0(M, \tilde{g})} \leq 2$ for every $j = 1 \dots n$.
- (3) $|a_i(x, \varphi)| \leq 2$ for every 1-form such that $\|\varphi\|_{C^0(M, \tilde{g})} \leq 1$.
- (4) $|b_j(x, v)| \leq 2$ for every vector field such that $\|v\|_{C^0(M, \tilde{g})} \leq 1$.

Observe that if we fix a point x , a normal coordinate system on a sufficiently small neighborhood of x is an almost Euclidean coordinate system.

Let $\{U_y \subset\subset M, \phi_y : U_y \rightarrow \mathbb{R}^n\}_{y \in M}$ be a family of almost Euclidean coordinate system indexed by $y \in M$. Let

$$(10) \quad \{U_\omega \subset\subset M, \phi_\omega : U_\omega \rightarrow \mathbb{R}^n\}_{\omega \in \Lambda}$$

be an almost Euclidean locally finite refinement of $\{U_y, \phi_y\}_{y \in M}$, where Λ is an index set. Denote the cardinality of Λ by $|\Lambda|$.

We denote the coordinate vector $\frac{\partial}{\partial x_i}$, $1 \leq i \leq n$, with respect to the coordinate system $\{U_\omega, \phi_\omega\}$ at the point $x \in M$ by $(\frac{\partial}{\partial x_i}(x))_\omega$. Its dual 1-form is denoted by $(dx_i(x))_\omega$.

Take a partition of unity $\{\psi_\omega : M \rightarrow \mathbb{R}\}_{\omega \in \Lambda}$ subordinated to $\{U_\omega\}_{\omega \in \Lambda}$. Consider the 1-forms $(\vartheta_i)_\omega : M \rightarrow T^*M$ defined by

$$(11) \quad (\vartheta_i(x))_\omega = (\psi(x))_\omega \cdot (dx_i(x))_\omega.$$

Notice that $(\vartheta_i)_\omega$ is a smooth 1-form on M for every i and ω . In the same fashion, we can take smooth vector fields $(u_i)_\omega : M \rightarrow T^*M$ defined by

$$(12) \quad (u_i(x))_\omega = (\psi(x))_\omega \cdot \left(\frac{\partial}{\partial x_i}(x) \right)_\omega.$$

PROPOSITION 3.10. *Considering the notation given before, we have the following formulas:*

$$\varphi(x) = \sum_{\omega \in \Lambda} \sum_{i=1}^n (\psi(x))_\omega \cdot (a_i(x, \varphi))_\omega \cdot (dx_i(x))_\omega$$

$$(13) \quad = \sum_{\omega \in \Lambda} \sum_{i=1}^n (a_i(x, \varphi))_{\omega} \cdot (\vartheta_i(x))_{\omega}$$

and

$$(14) \quad \begin{aligned} v(x) &= \sum_{\omega \in \Lambda} \sum_{j=1}^n (\psi(x))_{\omega} \cdot (b_j(x, v))_{\omega} \cdot \left(\frac{\partial}{\partial x_j} \right)_{\omega} \\ &= \sum_{\omega \in \Lambda} \sum_{j=1}^n (b_j(x, v))_{\omega} \cdot (u_j(x))_{\omega}. \end{aligned}$$

Proof. These formulas hold because the sum is locally finite. □

Let T be a tensor field of type (m, s) on M . Define the functions

$$(15) \quad T_{i_1 \dots i_m j_1 \dots j_s \omega_1 \dots \omega_{m+s}} = T((\vartheta_{i_1})_{\omega_1}, \dots, (\vartheta_{i_m})_{\omega_m}, (u_{j_1})_{\omega_{m+1}}, \dots, (u_{j_s})_{\omega_{m+s}}),$$

where the indexes i_1, \dots, i_m and j_1, \dots, j_s varies between 1 and n and the indexes $\omega_1, \dots, \omega_{m+s}$ are in Λ .

Proposition 3.11 define the function \bar{T} which will be useful afterwards.

PROPOSITION 3.11. *The function $\bar{T} : M \rightarrow \mathbb{R}$, defined by*

$$(16) \quad \bar{T}(x) = 2^{m+s} \sum_{(\omega_1, \dots, \omega_{m+s})} \sum_{(i_1, \dots, i_m)} \sum_{(j_1, \dots, j_s)} |T_{i_1 \dots i_m j_1 \dots j_s \omega_1 \dots \omega_{m+s}}(x)|$$

is greater than or equal to $T_{sup} : M \rightarrow \mathbb{R}$, which is given by

$$T_{sup}(x) = \sup_{\|\varphi_1\|_{C^0} \leq 1, \dots, \|u_s\|_{C^0} \leq 1} T(\varphi_1, \dots, u_s)(x).$$

Proof. The proof will be done for tensors of type $(1, 1)$. The general case follows similarly.

$$\begin{aligned} T(\varphi, v) &= T \left(\sum_{\omega_1} \sum_i (a_i(x, \varphi))_{\omega_1} (\vartheta_i(x))_{\omega_1}, \sum_{\omega_2} \sum_j (b_j(x, v))_{\omega_2} (u_j(x))_{\omega_2} \right) \\ &= \sum_i \sum_j \sum_{\omega_1, \omega_2} T((a_i(x, \varphi))_{\omega_1} (\vartheta_i(x))_{\omega_1}, (b_j(x, v))_{\omega_2} (u_j(x))_{\omega_2}) \\ &\leq 2^2 \sum_i \sum_j \sum_{\omega_1, \omega_2} |T_{ij\omega_1\omega_2}|, \end{aligned}$$

because $|a_i(x, \varphi)|$ and $|b_j(x, \varphi)|$ are less than or equal to 2. □

Now we define norms on $C^0(M, \tilde{g})$ and $L^p(M, \tilde{g})$ which make them complete normed spaces.

THEOREM 3.12. *Let (M, \tilde{g}) be a Riemannian manifold and $C^0(M, \tilde{g})$ the space of C^0 tensor fields of type (m, s) . Define $\|\cdot\|_{C^0(M, \tilde{g})} : C^0(M, \tilde{g}) \rightarrow \mathbb{R}$ by*

$$\|T\|_{C^0(M, \tilde{g})} = \sup_{\|\varphi_1\|_{C^0} \leq 1, \dots, \|v_s\|_{C^0} \leq 1} \|T(\varphi_1, \dots, v_s)\|_{C^0(M, \tilde{g})},$$

where φ_i , $1 \leq i \leq m$, and v_j , $1 \leq j \leq s$, are smooth 1-forms and smooth vector fields defined on M respectively. Then $\|\cdot\|_{C^0(M, \tilde{g})}$ is a complete norm on $C^0(M, \tilde{g})$.

Proof. It is straightforward to see that $\|\cdot\|_{C^0} := \|\cdot\|_{C^0(M, \tilde{g})}$ is a norm on $C^0(M, \tilde{g})$. Let us prove its completeness.

We will prove the completeness of $\|\cdot\|_{C^0}$ for tensor spaces of type $(1, 1)$. The general case follows similarly. So let us begin with a Cauchy sequence $\{T_l\}_{l \in \mathbb{N}}$. We will prove that there exists a tensor field T such that $\lim_{l \rightarrow \infty} \|T_l - T\|_{C^0} = 0$.

Take a locally finite covering of M as defined by (10). Define the functions $T_{ij\omega_1\omega_2} : M \rightarrow \mathbb{R}$ by

$$(17) \quad T_{ij\omega_1\omega_2} = \lim_{l \rightarrow \infty} T_l((\vartheta_i)_{\omega_1}, (u_j)_{\omega_2})$$

where $(\vartheta_i)_{\omega_1}$ and $(u_j)_{\omega_2}$ are given by (11) and (12) respectively. Define

$$(18) \quad T(\varphi, v) = \sum_{i,j=1}^n \left(\sum_{(\omega_1, \omega_2) \in \Lambda \times \Lambda} (a_i)_{\omega_1} \cdot (b_j)_{\omega_2} T_{ij\omega_1\omega_2} \right)$$

where $\varphi, v, (a_i)_{\omega_1}$ and $(b_j)_{\omega_2}$ are related by Eqs. (8) and (9). It is not difficult to see that T is a tensor of type $(1, 1)$. Let us prove that $\lim_{l \rightarrow \infty} \|T_l - T\|_{C^0} = 0$.

Let $\epsilon > 0$. Then there exists a $N_\epsilon \in \mathbb{N}$ such that if $k, l > N_\epsilon$, then $\|T_k - T_l\|_{C^0} < \epsilon/(32.n^2)$. This means that if we fix i, j, ω_1 and ω_2 , then

$$|T_{ij\omega_1\omega_2} - T_l((\vartheta_i(\cdot))_{\omega_1}, (u_j(\cdot))_{\omega_2})| \leq \frac{\epsilon}{8n^2} \psi_{\omega_1} \psi_{\omega_2}$$

for every $l > N_\epsilon$.

Let $\varphi \in T^*M$ and $v \in TM$ be a smooth 1-form and a smooth vector field respectively such that their C^0 norms are less than or equal to one. Then

$$\begin{aligned} & \|T(\varphi, v) - T_l(\varphi, v)\|_{C^0} \\ &= \left\| \sum_i \sum_j \sum_{\omega_1, \omega_2} (T - T_l)((a_i(x, \varphi))_{\omega_1} (\vartheta_i(x))_{\omega_1}, (b_j(x, v))_{\omega_2} (u_j(x))_{\omega_2}) \right\|_{C^0} \\ &\leq 4 \left\| \sum_i \sum_j \sum_{\omega_1, \omega_2} (T - T_l)((\vartheta_i(x))_{\omega_1}, (u_j(x))_{\omega_2}) \right\|_{C^0} \leq \frac{\epsilon}{2}. \end{aligned}$$

Thus

$$\|T - T_l\|_{C^0} = \sup_{\|\varphi\|_{C^0} \leq 1, \|v\|_{C^0} \leq 1} \|(T - T_l)(\varphi, v)\|_{C^0} \leq \frac{\epsilon}{2} < \epsilon$$

whenever $l > N_\epsilon$ and $\lim_{\epsilon \rightarrow 0} \|T - T_l\|_{C^0} = 0$. □

THEOREM 3.13. *Let (M, \tilde{g}) be a Riemannian manifold and $L^p(M, \tilde{g})$ the space of L^p tensor fields of type (m, s) . Define $\|\cdot\|_{L^p(M, \tilde{g})} : L^p(M, \tilde{g}) \rightarrow \mathbb{R}$ by*

$$\|T\|_{L^p(M, \tilde{g})} := \sup_{\|\varphi_1\|_{C^0} \leq 1, \dots, \|v_s\|_{C^0} \leq 1} \left(\int_M |T(\varphi_1, \dots, v_s)(y)|^p dV_{\tilde{g}}(y) \right)^{\frac{1}{p}},$$

where $\varphi_i, 1 \leq i \leq m$, and $v_j, 1 \leq j \leq s$, are smooth 1-forms and smooth vector fields defined on M respectively. Then $\|\cdot\|_{L^p(M, \tilde{g})}$ is a complete norm on $L^p(M, \tilde{g})$.

Proof. The proof will be done for tensor spaces of type $(1, 1)$. The general case follows similarly.

It is straightforward that the properties $\|T_1 + T_2\|_{L^p(M, \tilde{g})} \leq \|T_1\|_{L^p(M, \tilde{g})} + \|T_2\|_{L^p(M, \tilde{g})}$ and $\|c.T\|_{L^p(M, \tilde{g})} = |c| \cdot \|T\|_{L^p(M, \tilde{g})}$ holds for every $T, T_1, T_2 \in L^p(M, \tilde{g})$ and $c \in \mathbb{R}$.

Suppose that $\|T\|_{L^p(M, \tilde{g})} = 0$. Then $T = 0$ a.e.. In fact, take a locally finite covering $\{U_\omega, \phi_\omega\}_{\omega \in \Lambda}$ of M as defined by (10) and consider $(\vartheta_i)_\omega$ and $(u_j)_\omega$ as defined in (11) and (12) respectively. Then $T((\vartheta_i)_{\omega_1}, (u_j)_{\omega_2}) = 0$ a.e. for every $\{i, j, \omega_1, \omega_2\} \in \{1, \dots, n\} \times \{1, \dots, n\} \times \Lambda \times \Lambda$. But using (13) and (14), we can see that $T(\varphi, v) = 0$ a.e. for every smooth 1-form φ and every smooth vector field v . This implies that $T = 0$ a.e.. Therefore $\|\cdot\|_{L^p(M, \tilde{g})}$ is a norm defined on $L^p(M, \tilde{g})$.

In order to prove the completeness of $\|\cdot\|_{L^p(M, \tilde{g})}$, let us begin with a Cauchy sequence $\{T_l\}_{l \in \mathbb{N}}$. We will prove that it converges to the tensor field T defined by (18).

Claim 1: $T \in L^p(M, \tilde{g})$, that is, $\|T\|_{L^p(M, \tilde{g})} < \infty$.

The general idea here is to build a uniformly bounded sequence $\{Q_d\}_{d \in \mathbb{N}}$ of tensor fields in $L^p(M, \tilde{g})$ such that it converges pointwise to T as d goes to infinity. Then using Fatou's Lemma, we will be able to prove that $\|T\|_{L^p(M, \tilde{g})} < \infty$. The tensor field Q_d is essentially T restricted to a compact subset $K_d \subset M$, where $\{K_d\}_{d \in \mathbb{N}}$ is an increasing sequence of compact sets (that is, $K_d \subset K_{d+1}$ for every d) such that $\bigcup_{d \in \mathbb{N}} K_d = M$. The decomposition $T = Q_d + (T - Q_d)$ allow us to split T in a “compact part” Q_d and in its complement $T - Q_d$, what is typical in this kind of problem. Let us formalize the idea:

Let $\epsilon = \frac{1}{d} > 0$, where $d \in \mathbb{N}$. Then there exists a $N_d \in \mathbb{N}$ such that $\|T_{l_1} - T_{l_2}\|_{L^p(M, \tilde{g})} < \frac{\epsilon}{2}$ whenever $l_1, l_2 \geq N_d$.

Observe that there exists a compact subset $K_d \subset M$ such that

$$\|(T_{N_d})|_{(M-K_d)}\|_{L^p(M, \tilde{g})} < \frac{\epsilon}{2}.$$

In fact, if this is not the case, then it would not be difficult to build a vector field v and a 1-form φ such that $\|v\|_{C^0} \leq 1$, $\|\varphi\|_{C^0} \leq 1$ and $\|T_{N_d}(v, \varphi)\|_{L^p(M, \tilde{g})} > \|T_{N_d}\|_{L^p(M, \tilde{g})}$, which would give a contradiction. Then

$$(19) \quad \|(T_l)|_{(M-K_d)}\|_{L^p(M, \tilde{g})} < \epsilon$$

for every $l \geq N_d$. Moreover we can assume that $\{K_d\}_{d \in \mathbb{N}}$ is an increasing sequence of compact subsets of M such that $\bigcup_{d \in \mathbb{N}} K_d = M$.

Take a locally finite covering $\{U_\omega, \phi_\omega\}_{\omega \in \Lambda}$ of M as defined by (10) and a partition of unity $\{\psi_\omega : M \rightarrow \mathbb{R}\}_{\omega \in \Lambda}$ subordinated to $\{U_\omega\}_{\omega \in \Lambda}$. Consider $(\vartheta_i)_\omega$ and $(u_j)_\omega$ as defined in (11) and (12) respectively. Let $\{U_{\tilde{\omega}}\}_{\tilde{\omega} \in \Lambda_d} \subset \{U_\omega\}_{\omega \in \Lambda}$ be the set of coordinate neighborhoods such that $U_{\tilde{\omega}} \cap K_d \neq \emptyset$. It is not difficult to see that $\{U_{\tilde{\omega}}\}_{\tilde{\omega} \in \Lambda_d}$ is a finite set, because $\{U_\omega\}_{\omega \in \Lambda}$ is a locally finite covering of M . Observe also that $\Lambda_d \subset \Lambda_{d+1}$.

Let $T_{ij\omega_1\omega_2} : M \rightarrow \mathbb{R}$ be the functions defined by Eq. (17). Define the sequence of tensor fields $\{Q_d\}_{d \in \mathbb{N}}$ by

$$Q_d(\varphi, v) = \sum_{i,j=1}^n \left(\sum_{(\omega_1, \omega_2) \in \Lambda_d \times \Lambda_d} (a_i)_{\omega_1} \cdot (b_j)_{\omega_2} T_{ij\omega_1\omega_2} \right),$$

where φ is given by (13) and v is given by (14). Notice that $Q_d \rightarrow T$ as $d \rightarrow \infty$ everywhere.

The idea here is to prove that $\|Q_d\|_{L^p} \leq C$ for some fixed constant C and using the Fatou's Lemma we will be able to prove that $\|T\|_{L^p} \leq C$. Fix a 1-form φ and a vector field v such that $\|\varphi\|_{C^0}, \|v\|_{C^0} \leq 1$. We have the following estimate:

$$\begin{aligned}
& \|T_l(\varphi, v) - Q_d(\varphi, v)\|_{L^p(M, \tilde{g})} \\
& \leq \left\| \sum_{i,j=1}^n \left(\sum_{(\omega_1, \omega_2) \in \Lambda_d \times \Lambda_d} (a_i)_{\omega_1} \cdot (b_j)_{\omega_2} (T_l((\vartheta_i)_{\omega_1}, (u_j)_{\omega_2}) - T_{ij\omega_1\omega_2}) \right) \right\|_{L^p(M, \tilde{g})} \\
(20) \quad & + \left\| \sum_{i,j=1}^n \left(\sum_{(\omega_1, \omega_2) \notin \Lambda_d \times \Lambda_d} (a_i)_{\omega_1} \cdot (b_j)_{\omega_2} (T_l((\vartheta_i)_{\omega_1}, (u_j)_{\omega_2})) \right) \right\|_{L^p(M, \tilde{g})}.
\end{aligned}$$

Due to (19), the last term of the right-hand-side of (20) is less than ϵ for every $l \geq N_d$.

For the first term of the right-hand-side of (20), there exists $N'_d \geq N_d$ such that

$$\|(T_l((\vartheta_i)_{\omega_1}, (u_j)_{\omega_2}) - T_{ij\omega_1\omega_2})\|_{L^p(M, \tilde{g})} < \frac{\epsilon}{4 \cdot n^2 \cdot |\Lambda_d|^2}$$

for every $(i, j, \omega_1, \omega_2) \in \{1, \dots, n\} \times \{1, \dots, n\} \times \Lambda_d \times \Lambda_d$ and $l \geq N'_d$. Therefore the first term of the right-hand-side of (20) is less than ϵ for $l \geq N'_d$ what implies that the right-hand-side of (20) is less than 2ϵ .

These facts implies that

$$\begin{aligned}
& \| |Q_d(\varphi, v)|^p \|_{L^1(M, \tilde{g})}^{1/p} = \|Q_d(\varphi, v)\|_{L^p(M, \tilde{g})} \leq \sup_{l \geq N'_d} \|T_l(\varphi, v)\|_{L^p(M, \tilde{g})} + 2\epsilon \leq \\
& \sup_{l \in \mathbb{N}} \|T_l\|_{L^p(M, \tilde{g})} + 2 < \infty
\end{aligned}$$

for every $d \in \mathbb{N}$. Now we can use the Fatou's Lemma and the pointwise convergence $\lim_{d \rightarrow \infty} |Q_d(\varphi, v)|^p = |T(\varphi, v)|^p$ in order to conclude that $\| |T|^p \|_{L^1(M, \tilde{g})}^{1/p} = \|T\|_{L^p(M, \tilde{g})}$ is finite, what settles Claim 1.

Claim 2: The sequence $\{T_l\}_{l \in \mathbb{N}}$ converges to T in $L^p(M, \tilde{g})$.

Some ideas that we used to prove the finiteness of $\|T\|_{L^p(M, \tilde{g})}$ will appear here again.

For every $d' \in \mathbb{N}$, set $\epsilon = \frac{1}{d'} > 0$. Notice that $\{T - T_l\}_{l \in \mathbb{N}}$ is a Cauchy sequence in $L^p(M, \tilde{g})$. In the same way we did in Claim 1, there exist $N_{d'} \in \mathbb{N}$ and a compact set $K_{d'} \subset M$ such that

$$\|(T - T_l)|_{(M - K_{d'})}\|_{L^p(M, \tilde{g})} < \epsilon$$

for every $l \geq N_{d'}$ (Compare with (19)). We can take the sequence $\{K_d\}_{d \in \mathbb{N}}$ such that $K_d \subset K_{d+1}$ for every $d \in \mathbb{N}$. Denote by $\{U_{\tilde{\omega}}\}_{\tilde{\omega} \in \Lambda_{d'}} \subset \{U_{\omega}\}_{\omega \in \Lambda}$ the set of coordinate neighborhoods such that $U_{\tilde{\omega}} \cap K_{d'} \neq \emptyset$.

Let φ be a smooth 1-form such that $\|\varphi\|_{C^0} \leq 1$ and let v be a smooth vector field such that $\|v\|_{C^0} \leq 1$. Then

$$\left\| \sum_{i,j=1}^n \left(\sum_{(\omega_1, \omega_2) \notin \Lambda_{d'} \times \Lambda_{d'}} (a_i)_{\omega_1} \cdot (b_j)_{\omega_2} ((T - T_l)((\vartheta_i)_{\omega_1}, (u_j)_{\omega_2})) \right) \right\|_{L^p(M, \tilde{g})} < \epsilon.$$

Now observe that

$$\|T(\varphi, v) - T_l(\varphi, v)\|_{L^p(M, \tilde{g})}$$

$$\begin{aligned}
&\leq \left\| \sum_{i,j=1}^n \left(\sum_{(\omega_1, \omega_2) \in \Lambda_{d'} \times \Lambda_{d'}} (a_i)_{\omega_1} \cdot (b_j)_{\omega_2} ((T - T_l)((\vartheta_i)_{\omega_1}, (u_j)_{\omega_2})) \right) \right\|_{L^p(M, \tilde{g})} \\
&+ \left\| \sum_{i,j=1}^n \left(\sum_{(\omega_1, \omega_2) \notin \Lambda_{d'} \times \Lambda_{d'}} (a_i)_{\omega_1} \cdot (b_j)_{\omega_2} ((T - T_l)((\vartheta_i)_{\omega_1}, (u_j)_{\omega_2})) \right) \right\|_{L^p(M, \tilde{g})}.
\end{aligned}$$

Here we repeat the procedure we used to control the first term of the right-hand-side of (20): There exist $N'_{d'} \in \mathbb{N}$ such that

$$\left\| \sum_{i,j=1}^n \left(\sum_{(\omega_1, \omega_2) \in \Lambda_{d'} \times \Lambda_{d'}} (a_i)_{\omega_1} \cdot (b_j)_{\omega_2} ((T - T_l)((\vartheta_i)_{\omega_1}, (u_j)_{\omega_2})) \right) \right\|_{L^p(M, \tilde{g})} < \epsilon$$

for $l \geq N'_{d'}$. Observe that this inequality does not depend on v and φ . Then

$$\|T - T_l\|_{L^p(M, \tilde{g})} < 2\epsilon$$

for $l \geq N'_{d'}$. Therefore $\lim_{l \rightarrow \infty} T_l = T$ in $L^p(M, \tilde{g})$. □

The tensor field spaces $L^p_{\text{loc}}(M)$ and $C^0_{\text{loc}}(M)$ can be topologized in the same way as their correspondent function spaces:

DEFINITION 3.14. Let M be a differentiable manifold. We say that a sequence $\{T_i\}_{i \in \mathbb{N}}$ of tensor fields converges to T in $C^0_{\text{loc}}(M)$ if

$$\lim_{i \rightarrow \infty} \|(T_i)|_U - (T)|_U\|_{C^0(U, \tilde{g})} = 0$$

for every background metric \tilde{g} and every open set $U \subset\subset M$.

Analogously we say that a one-parameter family of tensor fields $\{T_\varepsilon\}_{\varepsilon > 0}$ converges to T in $C^0_{\text{loc}}(M)$ as ε goes to zero if

$$\lim_{\varepsilon \rightarrow 0} \|(T_\varepsilon)|_U - (T)|_U\|_{C^0(U, \tilde{g})} = 0$$

for every background metric \tilde{g} and every open set $U \subset\subset M$.

Observe that these convergences do not depend on the choice of \tilde{g} .

DEFINITION 3.15. Let M be a differentiable manifold. We say that a sequence $\{T_i\}_{i \in \mathbb{N}}$ of tensor fields converges to T in $L^p_{\text{loc}}(M)$ if

$$\lim_{i \rightarrow \infty} \|(T_i)|_U - (T)|_U\|_{L^p(U, \tilde{g})} = 0$$

for every background metric \tilde{g} and every open set $U \subset\subset M$.

Analogously we say that a one-parameter family of tensor fields $\{T_\varepsilon\}_{\varepsilon > 0}$ converges to T in $L^p_{\text{loc}}(M)$ as ε goes to zero if

$$\lim_{\varepsilon \rightarrow 0} \|(T_\varepsilon)|_U - (T)|_U\|_{L^p(U, \tilde{g})} = 0$$

for every background metric \tilde{g} and every open set $U \subset\subset M$.

Observe that these convergences do not depend on the choice of \tilde{g} .

4. THE MOLLIFIER SMOOTHING OF A NON-REGULAR TENSOR FIELD

Let us remember the definition of mollifier smoothing of a locally summable function $\widehat{f} : \widehat{U} \subset \mathbb{R}^n \rightarrow \mathbb{R}$.

Let $\widetilde{\eta} : \mathbb{R}^n \rightarrow \mathbb{R}$ be the C^∞ function defined by

$$\widetilde{\eta}(x) := \begin{cases} C \exp\left(\frac{1}{\|x\|^2-1}\right) & \text{if } \|x\| < 1 \\ 0 & \text{if } \|x\| \geq 1, \end{cases}$$

where C is a constant such that $\int_{\mathbb{R}^n} \widetilde{\eta}(x) dV = 1$. Now define

$$(21) \quad \bar{\eta}(x, y, \varepsilon) := \frac{1}{\varepsilon^n} \widetilde{\eta}\left(\frac{x-y}{\varepsilon}\right).$$

Let $U \subset\subset \widehat{U}$ and define the mollifier smoothing $\widehat{f}_\varepsilon : U \rightarrow \mathbb{R}^n$ of \widehat{f} by

$$\widehat{f}_\varepsilon(x) = \int_{\widehat{U}} \bar{\eta}(x, y, \varepsilon) \widehat{f}(y) dV(y)$$

where $\varepsilon < \text{dist}(U, \mathbb{R}^n - \widehat{U})$.

THEOREM 4.1. (1) \widehat{f}_ε is $C^\infty(U)$.

(2) $\widehat{f}_\varepsilon \rightarrow \widehat{f}$ a.e. as $\varepsilon \rightarrow 0$.

(3) If \widehat{f} is a continuous function, then $\widehat{f}_\varepsilon \rightarrow \widehat{f}$ uniformly on compact subsets of \widehat{U} .

(4) If $\widehat{f} \in L^p_{\text{loc}}(\widehat{U})$, then $\widehat{f}_\varepsilon \rightarrow \widehat{f}$ in $L^p_{\text{loc}}(\widehat{U})$.

Proof. See [7]. □

Let M be a differentiable manifold. Let us define the mollifier smoothing of a non-regular tensor field $\widehat{T} \in T^{m,s}M$.

When we define a mollifier smoothing of a function $\widehat{f} : \widehat{U} \subset \mathbb{R}^n \rightarrow \mathbb{R}$, the Euclidean metric on \widehat{U} takes part in the process. In order to define the mollifier smoothing of tensor fields on a differentiable manifold, we introduce the background metric \widetilde{g} which is a smooth Riemannian metric on M .

Let $U \subset\subset M$ be an open set (This includes the case $U = M$ if M is a closed differentiable manifold). Define the injectivity radius of U by

$$(22) \quad \text{inj}(U, \widetilde{g}) = \inf_{x \in (U, \widetilde{g})} \text{inj}(x)$$

where $\text{inj}(x)$ is the injectivity radius of $x \in (M, \widetilde{g})$. We say that a function $\eta : U \times M \times (0, \text{inj}(U, \widetilde{g}))$ is a mollifier if

(1) η is a smooth function.

(2) $\eta(x, \cdot, \varepsilon) : M \rightarrow \mathbb{R}$ has its support in $\bar{B}(x, \varepsilon)$.

(3) $\int_M \eta(x, y, \varepsilon) dV_{\widetilde{g}}(y) = 1$ for every $x \in U$ and $\varepsilon \in (0, \text{inj}(U, \widetilde{g}))$ fixed.

Observe that $\bar{\eta}$ defined by (21) satisfies the conditions above for the Euclidean case. Moreover we can adapt $\bar{\eta}$ for a Riemannian manifold in the following fashion: Define

$$\bar{\eta}(x, y, \varepsilon) = \begin{cases} \exp\left(\frac{1}{\left(\frac{\text{dist}(x, y)}{\varepsilon}\right)^2-1}\right) & \text{if } \text{dist}(x, y) < \varepsilon < \text{inj}(U, \widetilde{g}) \\ 0 & \text{if } \text{dist}(x, y) \geq \varepsilon. \end{cases}$$

The function $\check{\eta}$ is clearly C^∞ . Now define

$$(23) \quad \eta(x, y, \varepsilon) = \frac{\check{\eta}(x, y, \varepsilon)}{\int_M \check{\eta}(x, y, \varepsilon) dV_{\tilde{g}}(y)}.$$

It is not difficult to see that the function η defined above satisfies the properties of a mollifier on (M, \tilde{g}) and it will be called the *standard mollifier*. Moreover the following proposition holds:

PROPOSITION 4.2. *Let $\eta : U \times M \times (0, \text{inj}(U, \tilde{g})) \rightarrow \mathbb{R}$ be the standard mollifier. Then there exist a positive constant $\|\eta\|$ such that*

$$\eta(x, y, \varepsilon) \leq \frac{\|\eta\|}{\varepsilon^n},$$

where n is the dimension of M .

Proof. If we observe that

- (1) The proposition is trivially true for Euclidean spaces;
- (2) Every smooth Riemannian manifold is “locally almost Euclidean”;
- (3) It is enough to prove the proposition for small values of ε ;
- (4) \bar{U} is compact,

then it is not difficult to prove the proposition. □

Suppose that there exist a unique minimizing geodesic γ connecting $x \in (M, \tilde{g})$ and $y \in (M, \tilde{g})$. We denote the parallel transport between the tensor spaces $T_x^{m,s}M$ and $T_y^{m,s}M$ through γ by $\tilde{\tau}_{x,y}$. If we have $\xi \in T_x^{m,s}M$, then we can define a smooth tensor field $\xi(x; \tilde{g})$ on the geodesic ball $B(x, \varepsilon)$ with $\varepsilon < \text{inj}(x)$ using parallel transport by

$$\xi(x; \tilde{g})(y) := \tilde{\tau}_{x,y}(\xi), \quad y \in B(x, \varepsilon).$$

Let \hat{T} be a non-regular tensor field in $T^{m,s}M$. We define the mollifier smoothing \hat{T}_ε of \hat{T} with respect to \tilde{g} as follows:

DEFINITION 4.3. Let (M, \tilde{g}) be a differentiable manifold with a smooth background metric \tilde{g} . Consider an open set $U \subset\subset M$ and let $\varepsilon < \text{inj}(U, \tilde{g})$. We define the mollifier smoothing $\hat{T}_\varepsilon \in T^{m,s}U$ of a non-regular tensor field $\hat{T} \in T^{m,s}M$ with respect to \tilde{g} by

$$(24) \quad \hat{T}_\varepsilon(\varphi_1, \dots, \varphi_m, v_1, \dots, v_s)(x) = \int_{(B(x, \varepsilon), \tilde{g})} \eta(x, y, \varepsilon) \cdot \hat{T}(\varphi_1(x; \tilde{g}), \dots, \varphi_m(x; \tilde{g}), v_1(x; \tilde{g}), \dots, v_s(x; \tilde{g}))(y) \cdot dV_{\tilde{g}}(y).$$

where $\{\varphi_i \in T_x^*M; 1 \leq i \leq m\}$ and $\{v_i \in T_x M; 1 \leq i \leq s\}$.

REMARK 4.4. Sometimes it is more useful to see Definition (24) as

$$\begin{aligned} & \hat{T}_\varepsilon(\varphi_1, \dots, \varphi_m, v_1, \dots, v_s)(x) \\ &= \int_{(M, \tilde{g})} \eta(x, y, \varepsilon) \cdot \hat{T}(\varphi_1(x; \tilde{g}), \dots, \varphi_m(x; \tilde{g}), v_1(x; \tilde{g}), \dots, v_s(x; \tilde{g}))(y) \cdot dV_{\tilde{g}}(y) \end{aligned}$$

(Notice that $\eta(x, \cdot, \varepsilon)$ has its support in $B(x, \varepsilon)$).

THEOREM 4.5. *The function \hat{T}_ε defined in Eq. (24) is a smooth tensor field of type (m, s) on U . Moreover, if \hat{T} is a non-regular Riemannian metric on M , then \hat{T}_ε is a smooth Riemannian metric on M .*

Proof. In order to prove that \widehat{T}_ε is a tensor field of type (m, s) , it is enough to prove that it is linear on each variable, which is straightforward. If \widehat{T} is a non-regular Riemannian metric on M , then we have to prove that \widehat{T}_ε is symmetric and positive definite, which is also straightforward.

Let us prove the smoothness of \widehat{T}_ε . The tensor field \widehat{T}_ε is smooth if and only if the function $\widehat{T}_\varepsilon(\varphi_1, \dots, \varphi_m, v_1, \dots, v_s)$ is smooth for every family of smooth 1-forms $\{\varphi_i : M \rightarrow T^*M, 1 \leq i \leq m\}$ and every family of smooth vector fields $\{v_i : M \rightarrow TM, 1 \leq i \leq s\}$. Consider Definition (24) and notice that

- $\eta(\cdot, y, \varepsilon) : M \rightarrow \mathbb{R}$ is a smooth function.
- Fix $y \in M$. Due to Theorem 2.1, $v_i(\cdot; \widetilde{g})(y) =: B(y, \varepsilon) \rightarrow T_y M$ is smooth for every $1 \leq i \leq s$ (The smoothness of $\varphi_i(\cdot; \widetilde{g})(y)$ follows similarly).
- The partial derivatives with respect to x in (24) can be taken inside the integral infinitely because \widehat{T} is multilinear with respect to its variables.

Therefore \widehat{T}_ε is smooth, what settles the theorem. \square

We are interested to know when $\lim_{\varepsilon \rightarrow 0} \widehat{T}_\varepsilon = \widehat{T}$ is true. The theorem below states that this happens in several situations.

THEOREM 4.6. *Let \widehat{T} be a locally summable tensor field of type (m, s) on a Riemannian manifold (M, \widetilde{g}) . Then we have the following convergences:*

- (1) $\widehat{T}_\varepsilon \rightarrow \widehat{T}$ a.e. as $\varepsilon \rightarrow 0$.
- (2) If \widehat{T} is continuous, then $\widehat{T}_\varepsilon \rightarrow \widehat{T}$ in $C_{\text{loc}}^0(M)$ as $\varepsilon \rightarrow 0$.
- (3) If $1 \leq p < \infty$ and $\widehat{T} \in L_{\text{loc}}^p(M)$, then $\widehat{T}_\varepsilon \rightarrow \widehat{T}$ in $L_{\text{loc}}^p(M)$ as $\varepsilon \rightarrow 0$.

Proof.

- (1) $\widehat{T}_\varepsilon \rightarrow \widehat{T}$ a.e. as $\varepsilon \rightarrow 0$.

Let $B(x_0, r) \subset\subset (M, \widetilde{g})$ be a geodesic ball, with r less than the injectivity radius at x_0 . Let $\{\beta_1, \beta_2, \dots, \beta_n\}$ be a family of smooth one forms on $B(x_0, r)$ such that $\{\beta_1(x), \beta_2(x), \dots, \beta_n(x)\}$ is a basis of T_x^*M for every $x \in B(x_0, r)$. Analogously let $\{w_1, w_2, \dots, w_n\}$ be a family of smooth vector fields on $B(x_0, r)$ such that $\{w_1(x), w_2(x), \dots, w_n(x)\}$ is a basis of $T_x M$ for every $x \in B(x_0, r)$. Fix an m -tuple $(\beta_{k_1}, \beta_{k_2}, \dots, \beta_{k_m})$ and an s -tuple $(w_{j_1}, w_{j_2}, \dots, w_{j_s})$ (the elements of each family can eventually appear more than once). The Lebesgue's Differentiation Theorem states that

$$(25) \quad \lim_{\varepsilon \rightarrow 0} \frac{\int_{B(x, \varepsilon)} |\widehat{T}(\beta_{k_1}, \dots, \beta_{k_m})(y) - \widehat{T}(\beta_{k_1}, \dots, \beta_{k_m})(x)| \cdot dV_{\widetilde{g}}(y)}{\text{Vol}(B(x, \varepsilon))} = 0$$

for almost every $x \in B(x_0, r)$. Denote by L the set of points $x \in B(x_0, r)$ such that (25) holds for every pair of families $(\beta_{k_1}, \dots, \beta_{k_m})$ and $(w_{j_1}, \dots, w_{j_s})$ (L is the set of Lebesgue points of $\widehat{T}|_{B(x_0, r)}$). We have that $B(x_0, r) - L$ has measure zero.

We claim that $\widehat{T}_\varepsilon(x) \rightarrow \widehat{T}(x)$ as $\varepsilon \rightarrow 0$ if $x \in L$. In fact, consider a family $(\varphi_1, \dots, \varphi_m)$ of smooth 1-forms and a family (v_1, \dots, v_s) of smooth vector fields on M . Let $\varepsilon > 0$ such

that $B(x, \varepsilon) \subset B(x_0, r)$. It follows that

$$(26) \quad \left| \widehat{T}_\varepsilon(\varphi_1, \dots, v_s)(x) - \widehat{T}(\varphi_1, \dots, v_s)(x) \right| \\ = \left| \int_M \eta(x, y, \varepsilon) \left[\widehat{T}(\varphi_1(x; \tilde{g}), \dots, v_s(x; \tilde{g}))(y) - \widehat{T}(\varphi_1, \dots, v_s)(x) \right] dV_{\tilde{g}}(y) \right|.$$

For each point $y \in B(x, \varepsilon)$, we denote $\varphi_l(x; \tilde{g})(y) = \sum_{k=1}^n h_{lk}^*(y) \beta_k(y)$ and $v_i(x; \tilde{g})(y) = \sum_{j=1}^n h_{ij}(y) w_j(y)$, where $1 \leq l \leq m$ and $1 \leq i \leq s$. Observe that \tilde{h}_{lk}^* and \tilde{h}_{ij} are smooth functions for all the indexes l, k, i, j . Then we have that

$$(27) \quad \widehat{T}(\varphi_1(x; \tilde{g}), \dots, v_s(x; \tilde{g}))(y) \\ = \sum_{j_1=1}^n \dots \sum_{k_m=1}^n h_{1k_1}^*(y) \dots h_{mk_m}^*(y) h_{1j_1}(y) \dots h_{sj_s}(y) \widehat{T}(\beta_{k_1}, \dots, v_{j_s})(y)$$

for every $y \in B(x, \varepsilon)$ because \widehat{T} is multilinear. Therefore the integrand of the right-hand-side of Eq. (26) is a sum of terms like

$$\eta(x, y, \varepsilon) h_{1k_1}^*(y) \dots h_{mk_m}^*(y) h_{1j_1}(y) \dots h_{sj_s}(y) \widehat{T}(\beta_{k_1}, \dots, w_{j_s})(y) \\ - \eta(x, y, \varepsilon) h_{1k_1}^*(x) \dots h_{mk_m}^*(x) h_{1j_1}(x) \dots h_{sj_s}(x) \widehat{T}(\beta_{k_1}, \dots, w_{j_s})(x),$$

which, for brevity, we write as

$$\eta(x, y, \varepsilon) \widehat{T}^h(y) - \eta(x, y, \varepsilon) \widehat{T}^h(x).$$

Proposition 2.5 states that x is a Lebesgue point of \widehat{T}^h . Then (26) is less than or equal to the sum of terms like

$$\int_M |\eta(x, y, \varepsilon) [\widehat{T}^h(y) - \widehat{T}^h(x)]| dV_{\tilde{g}}(y) \leq \frac{\|\eta\|}{\varepsilon^n} \int_{B(x, \varepsilon)} |\widehat{T}^h(y) - \widehat{T}^h(x)| dV_{\tilde{g}}(y)$$

that goes to zero as ε goes to zero. Therefore $\widehat{T}_\varepsilon \rightarrow \widehat{T}$ for almost every $x \in B(x_0, r)$, what implies that $\widehat{T}_\varepsilon \rightarrow \widehat{T}$ a.e..

(2) If \widehat{T} is continuous, then $\widehat{T}_\varepsilon \rightarrow \widehat{T}$ in $C_{\text{loc}}^0(M)$ as $\varepsilon \rightarrow 0$.

We will prove the convergence for tensors T of type $(1, 1)$. The general case is analogous.

Fix an open set $U \subset\subset M$. Let $\{U_\omega \subset\subset M, \phi_\omega\}_{\omega \in \Lambda}$ be a locally finite covering of M by almost Euclidean coordinate systems. Take a partition of unity $\{\psi_\omega : M \rightarrow \mathbb{R}\}_{\omega \in \Lambda}$ subordinated to $\{U_\omega\}_{\omega \in \Lambda}$. Consider the functions $\widehat{T}_{ij\omega_1\omega_2}$ as in (15). Every $x \in M$ is a Lebesgue point of \widehat{T} what implies that $(\widehat{T}_\varepsilon)_{ij\omega_1\omega_2}(x) \rightarrow \widehat{T}_{ij\omega_1\omega_2}(x)$ as $\varepsilon \rightarrow 0$ for every $x \in U$. Then $(\widehat{T}_\varepsilon)_{ij\omega_1\omega_2}$ converges uniformly to $\widehat{T}_{ij\omega_1\omega_2}$ in U .

Set $\Lambda_U := \{\omega \in \Lambda; U_\omega \cap \bar{U} \neq \emptyset\}$. Observe that Λ_U is a finite index set. Fix $\epsilon > 0$ and let r_U be the injectivity radius of U in $(\bigcup_{\omega \in \Lambda_U} U_\omega, \tilde{g})$. For every $(i, j, \omega_1, \omega_2) \in \{1, \dots, n\} \times$

$\{1, \dots, n\} \times \Lambda_U \times \Lambda_U$, we can choose a family of positive numbers $\varepsilon_{ij\omega_1\omega_2} < r_U$ such that $\|(\widehat{T}_{\tilde{\varepsilon}_{ij\omega_1\omega_2}})_{ij\omega_1\omega_2} - \widehat{T}_{ij\omega_1\omega_2}\|_{C^0(U, \tilde{g})} < \frac{\epsilon}{4n^2|\Lambda_U|^2}$ for every $\tilde{\varepsilon}_{ij\omega_1\omega_2} < \varepsilon_{ij\omega_1\omega_2}$. If we choose $\tilde{\varepsilon} = \min_{ij\omega_1\omega_2} \varepsilon_{ij\omega_1\omega_2}$ we have that $\|\widehat{T}_\varepsilon - \widehat{T}\|_{C^0(U, \tilde{g})} < \epsilon$ for every $\varepsilon < \tilde{\varepsilon}$ (See (13), (14) and inequalities $|(a_i(x, \varphi))_{\omega_1}| \leq 2$, $|(b_j(x, v))_{\omega_2}| \leq 2$, $\|(\vartheta_i)_{\omega_1}\|_{C^0} \leq 1$ and $\|(u_j)_{\omega_2}\|_{C^0} \leq 1$). Therefore $\widehat{T}_\varepsilon \rightarrow \widehat{T}$ in $C_{\text{loc}}^0(M)$ as $\varepsilon \rightarrow 0$.

(3) If $1 \leq p < \infty$ and $\widehat{T} \in L^p_{\text{loc}}(M)$, then $\widehat{T}_\varepsilon \rightarrow \widehat{T}$ in $L^p_{\text{loc}}(M)$ as $\varepsilon \rightarrow 0$.

Fix an open set $U \subset\subset M$ and let $\{U_\omega \subset\subset M, \phi_\omega\}_{\omega \in \Lambda}$ be a locally finite covering of M by almost Euclidean coordinate systems. Denote $\Lambda_U = \{\omega \in \Lambda; U_\omega \cap \bar{U} \neq \emptyset\}$. Let $U_1 = \bigcup_{\omega \in \Lambda_U} U_\omega$ and denote $\Lambda_{U_1} = \{\omega \in \Lambda; U_\omega \cap \bar{U}_1 \neq \emptyset\}$.

Claim 1: $\|\widehat{T}_\varepsilon\|_{L^p(U, \tilde{g})} \leq C \cdot \|\widehat{T}\|_{L^p(U_1, \tilde{g})}$ for some positive constant C that does not depend on \widehat{T} and $\varepsilon < \text{inj}(U, (U_1, \tilde{g}))$.

Let $\{\varphi_1, \dots, \varphi_m\}$ and $\{v_1, \dots, v_s\}$ be respectively a family of smooth 1-forms and a family of smooth vector fields on M such that their $\|\cdot\|_{C^0(M, \tilde{g})}$ norm are less than or equal to 1. Let $1 < p < \infty$ (the case $p = 1$ is easier) and fix $x \in U$. Considering ε less than the injectivity radius of U in (U_1, \tilde{g}) , we have that

$$\begin{aligned} \left| \widehat{T}_\varepsilon(\varphi_1, \dots, v_s)(x) \right| &= \left| \int_{U_1} \eta(x, y, \varepsilon) \cdot \widehat{T}(\varphi_1(x; \tilde{g}), \dots, v_s(x; \tilde{g}))(y) \cdot dV_{\tilde{g}}(y) \right| \\ &\leq \int_{U_1} [\eta(x, y, \varepsilon)]^{1-\frac{1}{p}} \cdot [\eta(x, y, \varepsilon)]^{\frac{1}{p}} \cdot |\widehat{T}(\varphi_1(x; \tilde{g}), \dots, v_s(x; \tilde{g}))(y)| \cdot dV_{\tilde{g}}(y) \\ &\leq \left(\int_{U_1} \eta(x, y, \varepsilon) |\widehat{T}(\varphi_1(x; \tilde{g}), \dots, v_s(x; \tilde{g}))(y)|^p \cdot dV_{\tilde{g}}(y) \right)^{\frac{1}{p}}, \end{aligned}$$

where we used the Hölder inequality in the last step. Notice that this inequality is trivially true when $p = 1$. Integrating in x we have that

$$\begin{aligned} \|\widehat{T}_\varepsilon(\varphi_1, \dots, v_s)\|_{L^p(U, \tilde{g})}^p &= \int_U |\widehat{T}_\varepsilon(\varphi_1, \dots, v_s)(x)|^p \cdot dV_{\tilde{g}}(x) \\ &\leq \int_U \left(\int_{U_1} \eta(x, y, \varepsilon) |\widehat{T}(\varphi_1(x; \tilde{g}), \dots, v_s(x; \tilde{g}))(y)|^p \cdot dV_{\tilde{g}}(y) \right) dV_{\tilde{g}}(x) \\ &\leq \int_U \left(\int_{U_1} \eta(x, y, \varepsilon) |\bar{T}(y)|^p \cdot dV_{\tilde{g}}(y) \right) dV_{\tilde{g}}(x) \leq (*) \end{aligned}$$

where \bar{T} is defined by (16).

Using the Fubini's Theorem we have that

$$\begin{aligned} (*) &\leq \int_{U_1} \left(\int_U \eta(x, y, \varepsilon) |\bar{T}(y)|^p \cdot dV_{\tilde{g}}(x) \right) dV_{\tilde{g}}(y) \\ &\leq \int_{U_1} \left(\int_{B(y, \varepsilon) \cap U} \eta(x, y, \varepsilon) |\bar{T}(y)|^p \cdot dV_{\tilde{g}}(x) \right) dV_{\tilde{g}}(y) \\ &\leq \frac{\|\eta\|}{\varepsilon^n} \int_{U_1} \left(\int_{B(y, \varepsilon) \cap U} |\bar{T}(y)|^p dV_{\tilde{g}}(x) \right) dV_{\tilde{g}}(y) \\ &\leq \frac{\|\eta\|}{\varepsilon^n} \text{Vol}(B_{(\min K)}(\varepsilon)) \int_{U_1} |\bar{T}(y)|^p \cdot dV_{\tilde{g}}(y) \leq C^p \cdot \|\bar{T}\|_{L^p(U_1, \tilde{g})}^p \end{aligned}$$

where $(\min K)$ is the minimum sectional curvature on (U_1, \tilde{g}) , $\text{Vol}(B_{(\min K)}(\varepsilon))$ is the volume of the geodesic ball with radius ε and constant curvature $(\min K)$ and C is a constant that does not depend on $\varepsilon < \text{inj}(U, (U_1, \tilde{g}))$. Then

$$\|\widehat{T}_\varepsilon(\varphi_1, \dots, v_s)\|_{L^p(U, \tilde{g})} \leq C \cdot \|\bar{T}\|_{L^p(U_1, \tilde{g})}$$

$$\begin{aligned}
&\leq \left\| C.2^{m+s} \sum_{(\omega_1, \dots, \omega_{m+s})} \sum_{(i_1, \dots, i_m)} \sum_{(j_1, \dots, j_s)} |\widehat{T}_{i_1 \dots i_m j_1 \dots j_s \omega_1 \dots \omega_{m+s}}| \right\|_{L^p(U_1, \tilde{g})} \\
&\leq C.2^{m+s} \sum_{(\omega_1, \dots, \omega_{m+s})} \sum_{(i_1, \dots, i_m)} \sum_{(j_1, \dots, j_s)} \|\widehat{T}_{i_1 \dots i_m j_1 \dots j_s \omega_1 \dots \omega_{m+s}}\|_{L^p(U_1, \tilde{g})} \\
&\leq C.2^{m+s} n^{m+s} |\Lambda_{U_1}|^{m+s} \|\widehat{T}\|_{L^p(U_1, \tilde{g})}
\end{aligned}$$

which settles Claim 1.

$$\text{Claim 2: } \lim_{\varepsilon \rightarrow 0} \|\widehat{T} - \widehat{T}_\varepsilon\|_{L^p(U, \tilde{g})} = 0.$$

We will prove Claim 2 for tensor fields of type $(1, 1)$. The general case is analogous.

For every $i, j \in \{1, \dots, n\}$, $\omega_1, \omega_2 \in \Lambda_{U_1}$ and $\epsilon > 0$, there exist a continuous function $Q_{ij\omega_1\omega_2} : M \rightarrow \mathbb{R}$ such that

$$\|\widehat{T}_{ij\omega_1\omega_2} - Q_{ij\omega_1\omega_2}\|_{L^p(M, \tilde{g})} < \frac{\epsilon}{4|\Lambda_{U_1}|^2 n^2}.$$

Define

$$Q(\varphi, v) := \sum_{i,j=1}^n \left(\sum_{(\omega_1, \omega_2) \in \Lambda_{U_1} \times \Lambda_{U_1}} (a_i)_{\omega_1} (b_j)_{\omega_2} Q_{ij\omega_1\omega_2} \right),$$

where $(a_i)_{\omega_1}$ and $(b_j)_{\omega_2}$ are defined by Eqs. (8) and (9) respectively. It is easy to see that Q is a continuous tensor field, and it is straightforward that

$$\|\widehat{T} - Q\|_{L^p(U_1, \tilde{g})} < \epsilon.$$

Moreover we have that $\|\widehat{T}_\varepsilon - Q_\varepsilon\|_{L^p(U, \tilde{g})} < C\epsilon$ when ε is small enough (See Claim 1), what implies

$$\begin{aligned}
\|\widehat{T} - \widehat{T}_\varepsilon\|_{L^p(U, \tilde{g})} &\leq \|\widehat{T} - Q\|_{L^p(U, \tilde{g})} + \|Q - Q_\varepsilon\|_{L^p(U, \tilde{g})} + \|Q_\varepsilon - \widehat{T}_\varepsilon\|_{L^p(U, \tilde{g})} \\
&\leq \epsilon + \|Q - Q_\varepsilon\|_{L^p(U, \tilde{g})} + C.\epsilon.
\end{aligned}$$

Finally making ε even smaller, we have that $\|Q - Q_\varepsilon\|_{L^p(U, \tilde{g})} < \epsilon$ because $Q_\varepsilon \rightarrow Q$ uniformly on U . Therefore we have that

$$\lim_{\varepsilon \rightarrow 0} \|\widehat{T} - \widehat{T}_\varepsilon\|_{L^p(U, \tilde{g})} = 0,$$

what settles the theorem. □

5. THE MOLLIFIER SMOOTHING WITH RESPECT TO \mathcal{P}

Let M be a differentiable manifold and consider a non-regular tensor field \widehat{T} defined on M . In this section we define a mollifier smoothing $\widehat{T}_{\varepsilon, \mathcal{P}}$ of \widehat{T} that is quite adequate for our purposes: If \widehat{T} is a Riemannian metric of class C^2 , then the Levi-Civita connection and the Riemannian curvature tensor of $\widehat{T}_{\varepsilon, \mathcal{P}}$ converges to the Levi-Civita connection and to the Riemannian curvature tensor of \widehat{T} respectively as ε converges to zero. The idea is to take a locally finite covering M by open sets, each of them endowed with an Euclidean

background metric. We make the mollifier smoothing given by Eq. (24) on each open set of the covering and sum them weighted by a partition of unity subordinated to the covering. Let us formalize the idea.

Let M be a differentiable manifold. Let $\{(O_\omega, \tilde{e}_\omega)\}_{\omega \in \Lambda}$ and $\{U_\omega \subset\subset O_\omega\}_{\omega \in \Lambda}$ be families of open subsets of M indexed by $\omega \in \Lambda$ such that

- (1) O_ω is endowed with the Euclidean background metric \tilde{e}_ω for each $\omega \in \Lambda$.
- (2) $\{U_\omega\}_{\omega \in \Lambda}$ is a locally finite covering of M .
- (3) $\{x \in O_\omega; \text{dist}_{\tilde{e}_\omega}(x, U_\omega) < 1\} \subset\subset O_\omega$.

Denote this family by $\{(U_\omega \subset\subset O_\omega, \tilde{e}_\omega)\}_{\omega \in \Lambda}$. It is not difficult to see that such a family always exist.

Consider a non-regular tensor field $\hat{T} \in T^{m,s}M$. The tensor field $\hat{T}_{\omega\varepsilon} \in C^\infty(U_\omega)$ is defined as mollifier smoothing of $\hat{T}|_{O_\omega}$ in U_ω with respect to \tilde{e}_ω , that is

$$\begin{aligned} & \hat{T}_{\omega\varepsilon}(\varphi_1, \dots, \varphi_m, v_1, \dots, v_s)(x) \\ &= \int_{B(x, \varepsilon)} \eta(x, y, \varepsilon) \cdot \hat{T}(\varphi_1(x; \tilde{e}_\omega), \dots, v_s(x; \tilde{e}_\omega))(y) \cdot dV_{\tilde{e}_\omega}(y), \end{aligned}$$

where $x \in U_\omega$ and $\varepsilon < 1$. Now let us sum the tensor fields $\hat{T}_{\omega\varepsilon}$ using a partition of unity.

Let $\{\psi_\omega\}_{\omega \in \Lambda}$ be a partition of unity subordinated to $\{U_\omega\}_{\omega \in \Lambda}$. In order to simplify the notation, we denote the locally finite covering $\{(U_\omega \subset\subset O_\omega, \tilde{e}_\omega)\}_{\omega \in \Lambda}$ together with the partition of unity $\{\psi_\omega\}_{\omega \in \Lambda}$ by \mathcal{P} .

DEFINITION 5.1. Let \hat{T} be a non-regular tensor field of type (m, s) on a differentiable manifold M . Let \mathcal{P} be a locally finite covering $\{(U_\omega \subset\subset O_\omega, \tilde{g})\}_{\omega \in \Lambda}$ of M together with a partition of unity $\{\psi_\omega\}_{\omega \in \Lambda}$ subordinated to $\{(U_\omega \subset\subset O_\omega, \tilde{g})\}_{\omega \in \Lambda}$. The *mollifier smoothing of $\hat{T} \in T^{m,s}M$ with respect to \mathcal{P}* is a smooth tensor field $\hat{T}_{\varepsilon, \mathcal{P}}$ on M defined by

$$\hat{T}_{\varepsilon, \mathcal{P}}(x) = \sum_{\omega \in \Lambda} \psi_\omega(x) \hat{T}_{\omega\varepsilon}(x), \quad x \in M,$$

where $\varepsilon < 1$. It will be denoted simply by \hat{T}_ε if there is not any possibility of misunderstandings.

The mollifier smoothing with respect to \mathcal{P} has also nice convergence properties.

THEOREM 5.2. Let \hat{T} be a locally summable tensor field of type (m, s) on a differentiable manifold M . Let \mathcal{P} be a locally finite covering $\{(U_\omega \subset\subset O_\omega, \tilde{g})\}_{\omega \in \Lambda}$ of M together with a partition of unity $\{\psi_\omega\}_{\omega \in \Lambda}$ subordinated to $\{(U_\omega \subset\subset O_\omega, \tilde{g})\}_{\omega \in \Lambda}$. Then we have the following convergences:

- (1) $\hat{T}_{\varepsilon, \mathcal{P}} \rightarrow \hat{T}$ a.e. as $\varepsilon \rightarrow 0$.
- (2) If \hat{T} is continuous, then $\hat{T}_{\varepsilon, \mathcal{P}} \rightarrow \hat{T}$ in $C_{\text{loc}}^0(M)$ as $\varepsilon \rightarrow 0$.
- (3) If $1 \leq p < \infty$ and $\hat{T} \in L_{\text{loc}}^p(M)$, then $\hat{T}_{\varepsilon, \mathcal{P}} \rightarrow \hat{T}$ in $L_{\text{loc}}^p(M)$ as $\varepsilon \rightarrow 0$.

Proof. It is an immediate consequence of Theorem 4.6 and the fact that each open subset $U \subset\subset M$ intercepts only a finite number of elements of $\{U_\omega\}_{\omega \in \Lambda}$. □

Now we prove that the mollifier smoothing with respect to \mathcal{P} behaves well in the smooth case, that is, if \hat{g} is of class C^2 , then the first derivatives and the second derivatives of $\hat{g}_{\varepsilon, \mathcal{P}}$ converges to the correspondent derivatives of \hat{g} as ε goes to zero. In particular, the

Levi-Civita connection and the Riemannian curvature tensor of $(M, \widehat{g}_{\varepsilon, \mathcal{P}})$ converges to the Levi-Civita connection and the Riemannian curvature tensor of (M, \widehat{g}) as ε goes to zero.

REMARK 5.3. The Levi-Civita connection is not a tensor field. Then we can not apply fully the theory of non-regular tensor fields developed in the former sections. But some results about the convergence of Levi-Civita connections can be obtained. For instance we can think about the pointwise convergence of Levi-Civita connections as will be done in Theorem 5.4. Another application will be given in Section 7, where the parallel transport of a vector through a curve will be generalized for some non-regular Riemannian manifolds.

THEOREM 5.4. *Let \widehat{g} be a C^2 Riemannian metric defined on M and fix \mathcal{P} on M . Then the Levi-Civita connection and the Riemannian curvature tensor with respect to the metric $\widehat{g}_{\varepsilon, \mathcal{P}}$ converges everywhere to the the Levi-Civita connection and the Riemannian curvature tensor of (M, \widehat{g}) respectively as $\varepsilon \rightarrow 0$.*

Proof. Let \mathcal{P} denote the locally finite covering $\{(U_\omega \subset\subset O_\omega, \widetilde{e}_\omega)\}_{\omega \in \Lambda}$ with a partition of unity $\{\psi_\omega\}_{\omega \in \Lambda}$ subordinated to $\{U_\omega\}_{\omega \in \Lambda}$. We begin studying the convergence of $\widehat{g}_{\omega\varepsilon}$ to $\widehat{g}_\omega := \widehat{g}|_{U_\omega}$ at $x \in M$ up to derivatives of order two as ε goes to zero.

Fix $\omega \in \Lambda$, $x \in U_\omega$ and let (x_1, \dots, x_n) be a coordinate system in a neighborhood $N_\omega \subset U_\omega$ of x such that $(\widetilde{e}_\omega)_{ij} = \delta_{ij}$ in this coordinate system.

Denote the representation of \widehat{g} and $\widehat{g}_{\omega\varepsilon}$ in this coordinate system by $(\widehat{g}_\omega)_{ij}$ and $(\widehat{g}_{\omega\varepsilon})_{ij}$ respectively. Observe that

$$\begin{aligned} (\widehat{g}_{\omega\varepsilon})_{ij}(x) &= \widehat{g}_{\omega\varepsilon} \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) (x) \\ &= \int_M \eta(x, y, \varepsilon) \widehat{g}_\omega \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) (y) dV_{\widetilde{e}_\omega}(y) = \int_M \eta(x, y, \varepsilon) (\widehat{g}_\omega)_{ij}(y) dV_{\widetilde{e}_\omega}(y). \end{aligned}$$

Therefore all the first and second order derivatives of $(\widehat{g}_{\omega\varepsilon})_{ij}$ converges to the correspondent derivatives of $(\widehat{g}_\omega)_{ij}$ because N_ω can be considered as an open set in \mathbb{R}^n (see [7]).

Let (z_1, \dots, z_n) be another coordinate system in N_ω . The metric $\widehat{g}_{\omega\varepsilon}$ in this coordinate system is given by

$$(28) \quad (\widehat{g}_{\omega\varepsilon})_{\widehat{i}\widehat{j}}(x) = \sum_{i,j=1}^n (\widehat{g}_{\omega\varepsilon})_{ij}(x) \frac{\partial x_i}{\partial z_{\widehat{i}}} \frac{\partial x_j}{\partial z_{\widehat{j}}} = \sum_{i,j=1}^n \frac{\partial x_i}{\partial z_{\widehat{i}}} \frac{\partial x_j}{\partial z_{\widehat{j}}} \int_M \eta(x, y, \varepsilon) (\widehat{g}_\omega)_{ij}(y) dV_{\widetilde{e}_\omega}(y)$$

that converges to

$$\sum_{i,j=1}^n \frac{\partial x_i}{\partial z_{\widehat{i}}} \frac{\partial x_j}{\partial z_{\widehat{j}}} (\widehat{g}_\omega)_{ij}(x) = (\widehat{g}_\omega)_{\widehat{i}\widehat{j}}(x)$$

for every $\widehat{i}, \widehat{j} \in \{1, \dots, n\}$ as ε goes to 0.

Now let us study the convergence of the derivatives of $(\widehat{g}_{\omega\varepsilon})_{\widehat{i}\widehat{j}}$.

The coordinate vector fields satisfies the relationship

$$\frac{\partial}{\partial z_{\widehat{k}}} = \sum_{s=1}^n \frac{\partial x_s}{\partial z_{\widehat{k}}} \frac{\partial}{\partial x_s}.$$

Calculating the derivative of (28) with respect to $\partial/\partial z_{\widehat{k}}$, we have that

$$\frac{\partial}{\partial z_{\widehat{k}}} (\widehat{g}_{\omega\varepsilon})_{\widehat{i}\widehat{j}}(x)$$

$$\begin{aligned}
&= \sum_{i,j=1}^n \left[\frac{\partial}{\partial z_{\hat{k}}} \left(\frac{\partial x_i}{\partial z_{\hat{i}}} \frac{\partial x_j}{\partial z_{\hat{j}}} \right) \right] \int_M \eta(x, y, \varepsilon) (\widehat{g}_\omega)_{ij}(y) dV_{\widehat{e}_\omega}(y) \\
&+ \sum_{i,j,s=1}^n \left(\frac{\partial x_i}{\partial z_{\hat{i}}} \frac{\partial x_j}{\partial z_{\hat{j}}} \frac{\partial x_s}{\partial z_{\hat{k}}} \right) \left(\frac{\partial}{\partial x_s} \int_M \eta(x, y, \varepsilon) (\widehat{g}_\omega)_{ij}(y) dV_{\widehat{e}_\omega}(y) \right) \\
&= \sum_{i,j=1}^n \left[\frac{\partial}{\partial z_{\hat{k}}} \left(\frac{\partial x_i}{\partial z_{\hat{i}}} \frac{\partial x_j}{\partial z_{\hat{j}}} \right) \right] \int_M \eta(x, y, \varepsilon) (\widehat{g}_\omega)_{ij}(y) dV_{\widehat{e}_\omega}(y) \\
&+ \sum_{i,j,s=1}^n \left(\frac{\partial x_i}{\partial z_{\hat{i}}} \frac{\partial x_j}{\partial z_{\hat{j}}} \frac{\partial x_s}{\partial z_{\hat{k}}} \right) \left(\int_M \eta(x, y, \varepsilon) \frac{\partial}{\partial x_s} (\widehat{g}_\omega)_{ij}(y) dV_{\widehat{e}_\omega}(y) \right)
\end{aligned}$$

that converges to

$$\begin{aligned}
&\sum_{i,j=1}^n \left[\frac{\partial}{\partial z_{\hat{k}}} \left(\frac{\partial x_i}{\partial z_{\hat{i}}} \frac{\partial x_j}{\partial z_{\hat{j}}} \right) \right] (\widehat{g}_\omega)_{ij}(x) + \sum_{i,j,s=1}^n \left(\frac{\partial x_i}{\partial z_{\hat{i}}} \frac{\partial x_j}{\partial z_{\hat{j}}} \frac{\partial x_s}{\partial z_{\hat{k}}} \right) \frac{\partial}{\partial x_s} (\widehat{g}_\omega)_{ij}(x) \\
&= \frac{\partial}{\partial z_{\hat{k}}} (\widehat{g}_\omega)_{i\hat{j}}(x)
\end{aligned}$$

as ε goes to zero.

Analogously we have that

$$\frac{\partial}{\partial z_{\hat{i}}} \frac{\partial}{\partial z_{\hat{k}}} (\widehat{g}_{\omega\varepsilon})_{i\hat{j}}(x) \text{ converges to } \frac{\partial}{\partial z_{\hat{i}}} \frac{\partial}{\partial z_{\hat{k}}} (\widehat{g}_\omega)_{i\hat{j}}(x)$$

for every $\hat{i}, \hat{j}, \hat{l}, \hat{k} \in \{1, \dots, n\}$ as ε goes to zero.

Now let us study the convergence of $\widehat{g}_{\varepsilon, \mathcal{P}}$ to \widehat{g} at $x \in M$ up to derivatives of order two. The mollifier smoothing of \widehat{g} with respect to \mathcal{P} is given by the locally finite sum

$$\widehat{g}_{\varepsilon, \mathcal{P}} = \sum_{\omega} \psi_{\omega} \widehat{g}_{\omega\varepsilon}.$$

Take a coordinate system (z_1, \dots, z_n) in a neighborhood N_x of x . We have that

$$(\widehat{g}_{\varepsilon, \mathcal{P}})_{i\hat{j}} = \sum_{\omega} \psi_{\omega}(x) \cdot (\widehat{g}_{\omega\varepsilon})_{i\hat{j}}$$

in N_x , and using the calculations made before, it is not difficult to see that

$$\begin{aligned}
&(\widehat{g}_{\varepsilon, \mathcal{P}})_{i\hat{j}}(x) \text{ converges to } \widehat{g}_{i\hat{j}}(x), \\
&\frac{\partial}{\partial z_{\hat{k}}} (\widehat{g}_{\varepsilon, \mathcal{P}})_{i\hat{j}}(x) \text{ converges to } \frac{\partial}{\partial z_{\hat{k}}} \widehat{g}_{i\hat{j}}(x)
\end{aligned}$$

and that

$$\frac{\partial}{\partial z_{\hat{i}}} \frac{\partial}{\partial z_{\hat{k}}} (\widehat{g}_{\varepsilon, \mathcal{P}})_{i\hat{j}}(x) \text{ converges to } \frac{\partial}{\partial z_{\hat{i}}} \frac{\partial}{\partial z_{\hat{k}}} \widehat{g}_{i\hat{j}}(x)$$

as ε goes to zero.

Finally notice that the Christoffel symbols and the components Riemannian curvature tensor can be written in terms of the metric and its derivatives up to order two (See Eqs. (1) and (3)). Therefore the Christoffel symbols of $\widehat{g}_{\varepsilon, \mathcal{P}}$ converges to the Christoffel symbols of \widehat{g} and the components of the Riemannian curvature tensor of $\widehat{g}_{\varepsilon, \mathcal{P}}$ converges to the components of the Riemannian curvature tensor of \widehat{g} , as we wanted to prove.

□

We also have the following version of Theorem 5.4 for the κ -th Lipschitz-Killing curvature measures.

COROLLARY 5.5. *Let \widehat{g} be a C^2 Riemannian metric defined on M and fix \mathcal{P} on M . Then the κ -th Lipschitz Killing curvature measures of $(M, \widehat{g}_{\varepsilon, \mathcal{P}})$ converges everywhere to the κ -th Lipschitz Killing curvature measures of (M, \widehat{g}) .*

Proof. For every $\varepsilon > 0$, choose an orthonormal moving frame $\{w_{\varepsilon 1}, \dots, w_{\varepsilon n}\}$ for $(M, \widehat{g}_{\varepsilon, \mathcal{P}})$ such that $\{w_{\varepsilon 1}, \dots, w_{\varepsilon n}\}$ converges everywhere to an orthonormal moving frame $\{w_1, \dots, w_n\}$ of (M, \widehat{g}) as ε goes to zero. Consider the curvature forms of $(M, \widehat{g}_{\varepsilon, \mathcal{P}})$ and (M, \widehat{g}) with respect to these orthonormal frames. Then it follows from Theorem 5.4 and (4), (5) and (6) that the curvature forms of $(M, \widehat{g}_{\varepsilon, \mathcal{P}})$ converges everywhere to the respective curvature forms of (M, \widehat{g}) , what settles the corollary. □

Therefore if we want to generalize objects of the classical Riemannian geometry to non-regular Riemannian manifolds we can study how these objects behave when $(M, \widehat{g}_{\varepsilon, \mathcal{P}})$ converge to (M, \widehat{g}) as ε goes to zero. The following sections give applications of this type.

6. THE DEFINITION OF DISTANCE FOR NON-REGULAR RIEMANNIAN MANIFOLDS

Let M be a differentiable manifold. Two L^p_{loc} Riemannian metrics that differ on a subset of measure zero are identical. Then the distance between x and y in (M, \widehat{g}) can not be defined in the classical fashion, that is, as the infimum of the lengths of all piecewise regular curves that connects x and y . We should look for a more stable definition.

In this section we define the distance between two points in (M, \widehat{g}) using smooth approximations. But let us see some typical examples first. The mollifier smoothing with respect to a background metric \widetilde{g} and the mollifier smoothing with respect to \mathcal{P} provide smooth approximations of a non-regular Riemannian metric \widehat{g} . We use the former mollifier smoothing in order to see what happens if we try to define the distance between x and y in (M, \widehat{g}) as “ $d_{\widehat{g}}(x, y) = \lim_{\varepsilon \rightarrow 0} d_{\widehat{g}_{\varepsilon}}(x, y)$ ”.

EXAMPLE 6.1. Let (M, \widehat{g}) be a two-dimensional non-regular Riemannian manifold. Let $U \subset M$ be an open set, $\phi : U \rightarrow (-1, 1) \times (-1, 1) \subset \mathbb{R}^2$ a coordinate system and define a background Riemannian metric \widetilde{g} on M such that its restriction to U is the canonical Euclidean metric with respect to ϕ . Suppose that the metric \widehat{g} in this coordinate system is given by

$$(29) \quad \widehat{g} = \begin{cases} dx^2 + x^2 dy^2 & \text{if } x \neq 0 \\ dx^2 + dy^2 & \text{if } x = 0. \end{cases}$$

Observe that $\widehat{g}|_U$ is positive definite, but

$$\lim_{\varepsilon \rightarrow 0} d_{\widehat{g}_{\varepsilon}}((0, -1/2), (0, 1/2)) = 0.$$

We can see this fact representing (U, \widehat{g}) as the domain

$$\{(x, y) \in \mathbb{R}^2; x \in (-1, 1), -|x| < y < |x|\} \cup \{(0, 0)\}$$

with the canonical Euclidean metric.

EXAMPLE 6.2. Let $U \subset M$, \tilde{g} and ϕ as in Example 6.1. Suppose that the metric \hat{g} in the coordinate system ϕ is given by

$$(30) \quad \hat{g} = \begin{cases} dx^2 + dy^2 & \text{if } x = 0; \\ dx^2 + dy^2 & \text{if } \frac{1}{2^{2n+1}} < |x| < \frac{1}{2^{2n}}, n \in \mathbb{N} \cup \{0\}; \\ 2dx^2 + 2dy^2 & \text{if } \frac{1}{2^{2n+2}} < |x| < \frac{1}{2^{2n+1}}, n \in \mathbb{N} \cup \{0\}. \end{cases}$$

If we calculate the length of the straight line segment connecting $(0, -1/2)$ and $(0, 1/2)$ in (M, \hat{g}_ε) , then this length will oscillate as ε goes to zero. Therefore there is not any hope that the arc length of a curve converges always to a limit as ε goes to zero.

EXAMPLE 6.3. Let $U \subset M$, \tilde{g} and ϕ as in Example 6.1. Suppose that the metric \hat{g} in this coordinate system is given by

$$(31) \quad \hat{g} = \begin{cases} \frac{1}{\sqrt{x^2+y^2}}(dx^2 + dy^2) & \text{if } (x, y) \neq (0, 0) \\ dx^2 + dy^2 & \text{if } (x, y) = (0, 0). \end{cases}$$

Notice that $\hat{g} \in L_{\text{loc}}^p(M)$ for $p < 2$. For $P \neq (0, 0)$, we have that $d_{\hat{g}_\varepsilon}((0, 0), P) \rightarrow \infty$ as ε goes to zero, because \hat{g}_ε converges uniformly to \hat{g} on compact subsets of $(U - (0, 0))$ as ε goes to zero. This example shows that the distance induced by an $L_{\text{loc}}^p(M)$ Riemannian metric can be infinity.

Now let us define the Riemannian distance. Let (M, \hat{g}) be a non-regular Riemannian manifold. Denote by $\hat{g}(\varepsilon)$ a one-parameter family of smooth Riemannian metrics parametrized by $\varepsilon > 0$ such that $\lim_{\varepsilon \rightarrow 0} \hat{g}(\varepsilon) = \hat{g}$ in $L_{\text{loc}}^p(M)$. Consider

$$\mathcal{G} = \{\hat{g}(\varepsilon); \lim_{\varepsilon \rightarrow 0} \hat{g}(\varepsilon) = \hat{g} \text{ in } L_{\text{loc}}^p(M)\}$$

the collection of all one-parameter families of smooth Riemannian metrics parametrized by $\varepsilon > 0$ that converges to \hat{g} in $L_{\text{loc}}^p(M)$.

DEFINITION 6.4. Let (M, \hat{g}) be a non-regular Riemannian manifold. The Riemannian distance is defined by

$$d_{\hat{g}}(x, y) := \sup_{\hat{g}(\varepsilon) \in \mathcal{G}} [\limsup_{\varepsilon \rightarrow 0} d_{\hat{g}(\varepsilon)}(x, y)].$$

REMARK 6.5. If $\hat{g} \in C_{\text{loc}}^0(M)$, then our definition coincides with the classical definition of distance.

A non-regular Riemannian manifold (M, \hat{g}) with the Riemannian distance $d_{\hat{g}}$ is almost a metric space. Eventually the distance between two points can be zero (See Example 6.1). But the sets of points with zero distance are equivalence classes in M . The following theorem states that $d_{\hat{g}}$ induces a metric on the identification space M/\sim .

THEOREM 6.6. Let (M, \hat{g}) be a non-regular Riemannian manifold denote the identification space of M with the sets of points with zero distance identified by M/\sim . Then $d_{\hat{g}}$ induces a metric $\tilde{d}_{\hat{g}}$ on M/\sim by

$$\tilde{d}_{\hat{g}}(\tilde{x}, \tilde{y}) = d_{\hat{g}}(x, y)$$

where x and y are elements of the equivalence classes of \tilde{x} and \tilde{y} respectively.

Proof. Immediate. □

Notice that $(M/\sim, \tilde{d}_{\hat{g}})$ can be a metric space that is not homeomorphic to a differentiable manifold (See Example 6.1).

7. THE DEFINITION OF PARALLEL TRANSPORT ON NON-REGULAR RIEMANNIAN MANIFOLDS

Let (M, \hat{g}) be a non-regular Riemannian manifold. Let x and y be two points in M and consider a piecewise regular curve $\gamma : [a, b] \rightarrow M$ connecting x and y . Remember that $\hat{g}_{\varepsilon, \mathcal{P}}$ is the mollifier smoothing of \hat{g} with respect to \mathcal{P} . We denote by $\hat{\tau}_{x,y,\varepsilon}^{\gamma} : T_x^{m,s} M \rightarrow T_y^{m,s} M$ the parallel transport through γ with respect to the metric $\hat{g}_{\varepsilon, \mathcal{P}}$ (Here we do not carry \mathcal{P} in the notation of $\hat{\tau}_{x,y,\varepsilon}^{\gamma}$ for the sake of simplicity).

Now we are ready to define the parallel transport in non-regular Riemannian manifolds.

DEFINITION 7.1. Let (M, \hat{g}) be a non-regular Riemannian manifold. Let x and y be two points in M and consider a piecewise regular curve $\gamma : [a, b] \rightarrow M$ connecting x and y . The parallel transport $\hat{\tau}_{x,y}^{\gamma} : T_x^{m,s} M \rightarrow T_y^{m,s} M$ through γ is defined as

$$\hat{\tau}_{x,y}^{\gamma}(T) = \lim_{\varepsilon \rightarrow 0} \hat{\tau}_{x,y,\varepsilon}^{\gamma}(T)$$

if the limit exists and it does not depend on \mathcal{P} .

In this work, we will not look for general conditions that guarantees the existence of the parallel transport. Instead, we are going to see that the parallel transport can be defined through some curves on *piecewise smooth two-dimensional Riemannian manifolds* (See Definition 7.6 ahead). Before defining this kind of surface, we introduce some concepts and notations.

DEFINITION 7.2. Let M be a compact differentiable two-dimensional manifold (eventually with boundary). A *triangulation* of M is a homeomorphism $\Theta : \Upsilon \rightarrow M$ from a simplicial complex Υ onto M .

As usual, we call the image of each 0-dimensional simplex by vertex, the image of each 1-dimensional simplex by edge and the image of each 2-dimensional simplex by face (of the triangulation). The set of vertices will be denoted by V , the set of points which are in the interior of some edge will be denoted by E and the set of points which are in the interior of some face will be denoted by F . Then M can be written as the disjoint union $V \cup E \cup F$.

DEFINITION 7.3. Let M be a compact differentiable two-dimensional manifold (eventually with boundary) and $\Theta : \Upsilon \rightarrow M$ be a triangulation of M . We say that Θ is a *piecewise smooth triangulation* if Θ restricted to every two simplex is a diffeomorphism onto its image. In particular every edge is the image of a regular curve.

We want give the definition of piecewise smooth Riemannian metric on compact differentiable two-dimensional manifolds. The following example indicates what should happen on the edge of a triangulation.

EXAMPLE 7.4. Let $\Pi_1 = \{(x, y, z) \in \mathbb{R}^3; y \leq 0, z = 0\}$ and $\Pi_2 = \{(x, y, z) \in \mathbb{R}^3; y \geq 0, z = y\}$. Then $\Pi = \Pi_1 \cup \Pi_2$ is a smooth surface outside the x -axis. If we induce the canonical metric of \mathbb{R}^3 on the regular part of Π and put the coordinate system $(x, y) \in \mathbb{R}^2$ on Π , then the metric is given by

$$ds^2 = \begin{cases} dx^2 + dy^2 & \text{if } y < 0 \\ dx^2 + 2dy^2 & \text{if } y > 0. \end{cases}$$

Observe that the metric is not defined on the x -axis. This fact does not represent any problem because the edge is a set of measure zero. Although the metric can not be extended to the x -axis, it induces a metric dx^2 on it.

The behavior found in Example 7.4 is defined as follow.

DEFINITION 7.5. Let M^n be a differentiable manifold and let $M_1, M_2 \subset M$ be two disjoint open sets. Suppose that $M_3^{n-1} \subset \bar{M}_1 \cap \bar{M}_2$ is a differentiable $(n-1)$ -dimensional submanifold of M , where \bar{M}_1 and \bar{M}_2 denote the closure of M_1 and M_2 in M respectively. Let g_1 and g_2 be two smooth Riemannian metrics on M_1 and M_2 respectively such that they induces smooth Riemannian metrics on M_3 . We say that \bar{M}_1 and \bar{M}_2 *glues nicely through M_3* if the Riemannian metric induced on M_3 by g_1 is equal to the Riemannian metric induced by g_2 .

We are now ready to give the definition of piecewise smooth two-dimensional Riemannian manifold.

DEFINITION 7.6. Let (M, \hat{g}) be a compact two-dimensional non-regular Riemannian manifold (eventually with boundary). We say that \hat{g} is a piecewise smooth Riemannian metric if there exist a piecewise smooth triangulation $\Theta : \Upsilon \rightarrow M$ such that

- (1) $\hat{g}|_{\Theta(\text{int}(\Xi))}$ is smooth for every two simplex $\Xi \in C$ and $\hat{g}|_{\Theta(\text{int}(\Xi))}$ is smoothly extendable to $\Theta(\Xi)$;
- (2) If Ξ_1 and Ξ_2 are simplexes which have an edge in common, then $(\text{int}(\Xi_1), \hat{g})$ and $(\text{int}(\Xi_2), \hat{g})$ glues nicely through $E \cap \Xi_1 \cap \Xi_2$.

In this case, we say that Θ is a *triangulation associated to (M, \hat{g})* .

A compact differentiable two-dimensional manifold (eventually with boundary) with a piecewise smooth Riemannian metric is called a *piecewise smooth two-dimensional Riemannian manifold*.

REMARK 7.7. Observe that \hat{g} does not need to satisfy any condition over V . Formally it does not represent any problem because it is enough to define a non-regular Riemannian metric outside a set of measure zero. Geometrically, it is quite natural to do so because the vertices of a polyhedral surface (eventually with non-flat faces and edges) are quite wild compared to its regular part. What is important here is that all the geometrical information of a vertex is found on its neighborhood.

Here we begin to study the parallel transport in a piecewise smooth two-dimensional Riemannian manifold (M, \hat{g}) . The following theorem tell us that when γ crosses an edge transversally, then the parallel transport of a vector through γ keep its angle with the edge.

THEOREM 7.8. Let (M, \hat{g}) be a piecewise smooth two-dimensional Riemannian manifold and fix \mathcal{P} . Let $\nu : [a, b] \rightarrow E$ be a regular parametrization of a piece of edge. Let $\gamma : [-\epsilon, \epsilon] \rightarrow M$ be a regular curve such that it intersects $\nu([a, b])$ transversally at the point $\gamma(0) = \nu(0)$. Suppose also that $(E \cup V) \cap \gamma([-\epsilon, \epsilon]) = \gamma(0)$. Let (t, s) be a coordinate system of a neighborhood of $\gamma([-\epsilon, \epsilon])$ such that $\gamma(t)$ has coordinates $(t, 0)$ and $\nu(s)$ has coordinates $(0, s)$. Fix $v \in T_{\gamma(-\epsilon)}M$. Then the parallel transport $\hat{\tau}_{x,y}^\gamma$ through γ with respect to the metric \hat{g} is well defined and

$$\lim_{t \rightarrow 0^-} \text{angle} \left(\frac{\partial}{\partial s}(t, 0), \hat{\tau}_{\gamma(-\epsilon), \gamma(t)}^\gamma(v) \right) = \lim_{t \rightarrow 0^+} \text{angle} \left(\frac{\partial}{\partial s}(t, 0), \hat{\tau}_{\gamma(-\epsilon), \gamma(t)}^\gamma(v) \right).$$

Proof. First of all we remark that ε and ϵ are different variables.

Let $v_\varepsilon(t)$ be the parallel transport of v through γ with respect to the metric $\widehat{g}_\varepsilon = \widehat{g}_{\varepsilon, \mathcal{P}}$. Let $\cos_\varepsilon(t)$ be the cosine of the angle between $v_\varepsilon(t)$ and $\frac{\partial}{\partial s}(t, 0)$.

If $\widehat{g}_{\varepsilon ij}$ are the components of \widehat{g}_ε with respect to (t, s) we have that

$$\cos_\varepsilon(t) = \frac{v_{\varepsilon 1}(t) \cdot \widehat{g}_{\varepsilon 12}(t) + v_{\varepsilon 2}(t) \cdot \widehat{g}_{\varepsilon 22}(t)}{\sqrt{\widehat{g}_{\varepsilon 11}(t) \cdot v_{\varepsilon 1}^2(t) + 2 \cdot \widehat{g}_{\varepsilon 12}(t) \cdot v_{\varepsilon 1}(t) \cdot v_{\varepsilon 2}(t) + \widehat{g}_{\varepsilon 22}(t) \cdot v_{\varepsilon 2}^2(t)} \cdot \sqrt{\widehat{g}_{\varepsilon 22}(t)}}$$

where $v_{\varepsilon 1}(t)$ and $v_{\varepsilon 2}(t)$ are the components of v_ε with respect to (t, s) . Now we use Eq. (1) and

$$\nabla_{\frac{\partial}{\partial t}} v_{i\varepsilon} = -(\Gamma_{1i}^1)_\varepsilon v_{\varepsilon 1}(t) - (\Gamma_{1i}^2)_\varepsilon v_{\varepsilon 2}(t)$$

in order to get

$$(32) \quad \frac{\partial}{\partial t} \cos_\varepsilon(t) = \frac{1}{2} \frac{v_{\varepsilon 1}(t) \left(\frac{\partial}{\partial s} \widehat{g}_{\varepsilon 11}(t) \cdot \widehat{g}_{\varepsilon 22}(t) - \frac{\partial}{\partial t} \widehat{g}_{\varepsilon 22}(t) \cdot \widehat{g}_{\varepsilon 12}(t) \right)}{\sqrt{\widehat{g}_{\varepsilon 11}(t) \cdot v_{\varepsilon 1}^2(t) + 2 \cdot \widehat{g}_{\varepsilon 12}(t) \cdot v_{\varepsilon 1}(t) \cdot v_{\varepsilon 2}(t) + \widehat{g}_{\varepsilon 22}(t) \cdot v_{\varepsilon 2}^2(t)} \cdot (\widehat{g}_{\varepsilon 22}(t))^{\frac{3}{2}}}.$$

The term $\frac{\partial}{\partial s} \widehat{g}_{\varepsilon 11}(t)$ is uniformly bounded with respect to ε because the derivation is done parallel to the edge. The term $\frac{\partial}{\partial t} \widehat{g}_{\varepsilon 22}(t)$ is uniformly bounded in terms of ε because the faces glue nicely through the edge and \widehat{g}_{22} is a Lipschitz function. Then the total variation of $\cos_\varepsilon(t)$ goes to zero as ϵ goes to zero. This settles the theorem. \square

THEOREM 7.9. *Let (M, \widehat{g}) be a piecewise smooth two-dimensional Riemannian manifold. Let $\gamma : [a, b] \rightarrow M$ be a piecewise regular curve such that it does not intercept V and it intercepts E transversally. Then the parallel transport $\hat{\tau}_{x,y}^\gamma : T_{\gamma(a)}^{m,s} M \rightarrow T_{\gamma(b)}^{m,s} M$ through γ is well defined.*

Proof. The parallel transport of a scalar is trivial. The parallel transport of a vector through $\gamma|_F$ is well defined due to Theorem 5.4 and the parallel transport in a neighborhood of E is well defined due to Theorem 7.8. The parallel transport of 1-forms through γ is well defined: In fact, given a $\varphi \in T_{\gamma(a)}^* M$, there exist a unique field of 1-forms through γ such that its contraction with parallel vector fields is constant through γ . It is not difficult to see that this field of 1-forms are the parallel transport of φ through γ . Finally the parallel transport of other types of tensors through γ is also well defined: Analogously to the case of 1-forms, it is not difficult to prove that the parallel transport of general tensors coincide with sums of tensor products of parallel vector fields and parallel 1-forms. \square

8. THE LIPSCHITZ-KILLING CURVATURE MEASURE FOR CLOSED NON-REGULAR RIEMANNIAN MANIFOLDS

In this section, we give another application of the mollifier smoothing with respect to \mathcal{P} . We generalize the κ -th Lipschitz-Killing curvature measure for closed non-regular Riemannian manifolds using the mollifier smoothing $\widehat{g}_{\varepsilon, \mathcal{P}}$. It will be a signed measure $\mathcal{R}^\kappa(\cdot, \widehat{g}) : \widetilde{\mathcal{B}}(M) \rightarrow \mathbb{R}$, where $\widetilde{\mathcal{B}}(M)$ is a σ -algebra that contains all Borel sets of M .

Denote the κ -th Lipschitz-Killing curvature measure (defined on the open subsets of M) of $(M, \widehat{g}_{\varepsilon, \mathcal{P}})$ by $\mathcal{R}^\kappa(\cdot, \widehat{g}_{\varepsilon, \mathcal{P}})$. If we want to extend the the Lipschitz-Killing curvature measure $\mathcal{R}^\kappa(\cdot, \widehat{g})$ to a closed non-regular Riemannian manifold (M, \widehat{g}) , the first natural trial could be:

“DEFINITION”. The *Lipschitz-Killing curvature measure* of an open set O with respect to \widehat{g} is defined by

$$\mathcal{R}^\kappa(O, \widehat{g}) := \lim_{\varepsilon \rightarrow 0} \mathcal{R}^\kappa(O, \widehat{g}_{\varepsilon, \mathcal{P}})$$

if this limit exists and it does not depend on \mathcal{P} .

Unfortunately this approach does not work, even for the example given in the introduction where a two dimensional sphere converges to a cube. In fact, consider the open set $M - \{p\}$ where p is a vertex of the cube. Remember that $\mathcal{R}^2(\cdot, \widehat{g}_{\varepsilon, \mathcal{P}})$ is proportional to the Gaussian curvature measure $\mathcal{K}(\cdot, \widehat{g}_{\varepsilon, \mathcal{P}})$. The “definition” above give us $\mathcal{K}(M - \{p\}, \widehat{g}) = 4\pi$ while the “right answer” should be $\mathcal{K}(M - \{p\}, \widehat{g}) = 7\pi/2$. The problem here is that the curvature of $\{p\}$ has its influence in the mollifier smoothing. If we want to calculate $\mathcal{R}^\kappa(M - \{p\}, \widehat{g})$ “correctly”, we should separate $\{p\}$ from the calculations. This can be done as follows:

Denote the topology of M by $\mathcal{T}(M)$, the σ -algebra of Borel sets of M by $\mathcal{B}(M)$ and the collection of all subsets of M by 2^M .

DEFINITION 8.1. Let (M, \widehat{g}) be a closed non-regular Riemannian manifold. Fix $\kappa \in \{1, \dots, n\}$. Let $\mathcal{A}(M) \subset \mathcal{T}(M)$ be a family of open sets such that

- (1) $\mathcal{A}(M)$ contains a basis of the topology $\mathcal{T}(M)$;
- (2) For every $O \in \mathcal{T}(M)$, there exist a increasing sequence of open sets $\{\widetilde{O}_i\}_{i \in \mathbb{N}}$ in $\mathcal{A}(M)$ such that $\widetilde{O}_i \subset \subset O$ for every $i \in \mathbb{N}$ and $\cup_{i=1}^\infty \widetilde{O}_i = O$;
- (3)

$$\mathcal{R}^\kappa(\widetilde{O}, \widehat{g}) := \lim_{\varepsilon \rightarrow 0} \mathcal{R}^\kappa(\widetilde{O}, \widehat{g}_{\varepsilon, \mathcal{P}})$$

exists for every $\widetilde{O} \in \mathcal{A}(M)$ and this limit does not depend on \mathcal{P} ;

- (4) $\mathcal{R}^\kappa(\cdot, \widehat{g}) : \mathcal{A}(M) \rightarrow \mathbb{R}$ can be decomposed as

$$\mathcal{R}^\kappa(\cdot, \widehat{g}) = \mathcal{R}^{\kappa+}(\cdot, \widehat{g}) - \mathcal{R}^{\kappa-}(\cdot, \widehat{g})$$

where $\mathcal{R}^{\kappa+}(\cdot, \widehat{g}) : \mathcal{A}(M) \rightarrow \mathbb{R}$ and $\mathcal{R}^{\kappa-}(\cdot, \widehat{g}) : \mathcal{A}(M) \rightarrow \mathbb{R}$ are two non-negative functions;

- (5) Let $\{\widetilde{O}_j\}_{j \in \mathbb{N}}$ be a sequence in $\mathcal{A}(M)$. Denote $U_i = \cup_{j=i}^\infty \widetilde{O}_j$. If $\cap_{i=1}^\infty U_i = \emptyset$, then

$$\lim_{i \rightarrow \infty} \mathcal{R}^{\kappa+}(\widetilde{O}_i, \widehat{g}) = \lim_{i \rightarrow \infty} \mathcal{R}^{\kappa-}(\widetilde{O}_i, \widehat{g}) = 0;$$

- (6) If $\widetilde{O}_1, \widetilde{O}_2, \widetilde{O}_3, \widetilde{O}_4 \in \mathcal{A}(M)$ are such that $\widetilde{O}_1 \cup \widetilde{O}_2 \subset \widetilde{O}_3 \cup \widetilde{O}_4$ and $\widetilde{O}_1 \cap \widetilde{O}_2 \subset \widetilde{O}_3 \cap \widetilde{O}_4$, then

$$\mathcal{R}^{\kappa+}(\widetilde{O}_1, \widehat{g}) + \mathcal{R}^{\kappa+}(\widetilde{O}_2, \widehat{g}) \leq \mathcal{R}^{\kappa+}(\widetilde{O}_3, \widehat{g}) + \mathcal{R}^{\kappa+}(\widetilde{O}_4, \widehat{g})$$

and

$$\mathcal{R}^{\kappa-}(\widetilde{O}_1, \widehat{g}) + \mathcal{R}^{\kappa-}(\widetilde{O}_2, \widehat{g}) \leq \mathcal{R}^{\kappa-}(\widetilde{O}_3, \widehat{g}) + \mathcal{R}^{\kappa-}(\widetilde{O}_4, \widehat{g});$$

- (7) If $\{\widetilde{O}_i\}_{i \in \mathbb{N}}$ is an increasing sequence in $\mathcal{A}(M)$ such that $\cup_{i=1}^\infty \widetilde{O}_i = \widetilde{O} \in \mathcal{A}(M)$ and $\widetilde{O}_i \subset \subset \widetilde{O}$, then

$$\lim_{i \rightarrow \infty} \mathcal{R}^{\kappa+}(\widetilde{O}_i, \widehat{g}) = \mathcal{R}^{\kappa+}(\widetilde{O}, \widehat{g})$$

and

$$\lim_{i \rightarrow \infty} \mathcal{R}^{\kappa-}(\widetilde{O}_i, \widehat{g}) = \mathcal{R}^{\kappa-}(\widetilde{O}, \widehat{g}).$$

Then the triple $(\mathcal{A}(M), \mathcal{R}^{\kappa+}(\cdot, \widehat{g}), \mathcal{R}^{\kappa-}(\cdot, \widehat{g}))$ is called a κ -th *Lipschitz-Killing curvature measure generator*.

As we said before, a κ -th Lipschitz-Killing curvature measure generator can be thought as a set of “regular” open sets with their respective total κ -th Lipschitz-Killing curvature. Observe that all the properties given in Definition 8.1 must be satisfied if we want that $\mathcal{R}^\kappa(\cdot, \widehat{g})$ is extendable to a signed measure on $\mathcal{B}(M)$.

REMARK 8.2. The notation $\mathcal{R}^{\kappa^\pm}(\cdot, \widehat{g})$ will be used in order to represent $\mathcal{R}^{\kappa^+}(\cdot, \widehat{g})$ and $\mathcal{R}^{\kappa^-}(\cdot, \widehat{g})$ simultaneously.

Let (M, \widehat{g}) be a closed non-regular Riemannian manifold endowed with a κ -th Lipschitz-Killing curvature measure generator $(\mathcal{A}(M), \mathcal{R}^{\kappa^+}(\cdot, \widehat{g}), \mathcal{R}^{\kappa^-}(\cdot, \widehat{g}))$. Our aim is to extend $\mathcal{R}^{\kappa^\pm}(\cdot, \widehat{g})$ for $\mathcal{B}(M)$.

The first step is to extend $\mathcal{R}^{\kappa^\pm}(\cdot, \widehat{g})$ to $\mathcal{T}(M)$. In order to do it, we prove some lemmas:

LEMMA 8.3. *Let (M, \widehat{g}) be a closed non-regular Riemannian manifold with a κ -th Lipschitz-Killing curvature measure generator $(\mathcal{A}(M), \mathcal{R}^{\kappa^+}(\cdot, \widehat{g}), \mathcal{R}^{\kappa^-}(\cdot, \widehat{g}))$. Let $\widetilde{O}, \widetilde{U} \in \mathcal{A}(M)$. Suppose that $\widetilde{O} \subset \widetilde{U}$. Then*

$$\mathcal{R}^{\kappa^\pm}(\widetilde{O}, \widehat{g}) \leq \mathcal{R}^{\kappa^\pm}(\widetilde{U}, \widehat{g}).$$

Proof. Take $\widetilde{O}_1 = \widetilde{O}$, $\widetilde{O}_2 = \widetilde{O}$, $\widetilde{O}_3 = \widetilde{U}$ and $\widetilde{O}_4 = \widetilde{O}$ and the lemma is an immediate consequence of Property (6) of Definition 8.1. \square

DEFINITION 8.4. Let (M, \widehat{g}) be a closed non-regular Riemannian manifold with a Lipschitz-Killing curvature measure generator $(\mathcal{A}(M), \mathcal{R}^{\kappa^+}(\cdot, \widehat{g}), \mathcal{R}^{\kappa^-}(\cdot, \widehat{g}))$. We define the Lipschitz-Killing curvature measure of $O \in \mathcal{T}(M)$ by

$$\mathcal{R}^\kappa(O, \widehat{g}) = \mathcal{R}^{\kappa^+}(O, \widehat{g}) - \mathcal{R}^{\kappa^-}(O, \widehat{g})$$

where

$$(33) \quad \mathcal{R}^{\kappa^\pm}(O, \widehat{g}) := \sup_{\widetilde{O} \in \mathcal{A}(M), \widetilde{O} \subset \subset O} \mathcal{R}^{\kappa^\pm}(\widetilde{O}, \widehat{g}).$$

LEMMA 8.5. *Let (M, \widehat{g}) be a closed non-regular Riemannian manifold with a Lipschitz-Killing curvature measure generator $(\mathcal{A}(M), \mathcal{R}^{\kappa^+}(\cdot, \widehat{g}), \mathcal{R}^{\kappa^-}(\cdot, \widehat{g}))$. Let O_1 and O_2 be two open sets of M . Then $\mathcal{R}^{\kappa^\pm}(O_1 \cup O_2, \widehat{g}) + \mathcal{R}^{\kappa^\pm}(O_1 \cap O_2, \widehat{g}) = \mathcal{R}^{\kappa^\pm}(O_1, \widehat{g}) + \mathcal{R}^{\kappa^\pm}(O_2, \widehat{g})$.*

Proof.

Claim 1: $\mathcal{R}^{\kappa^\pm}(O_1, \widehat{g}) + \mathcal{R}^{\kappa^\pm}(O_2, \widehat{g}) \leq \mathcal{R}^{\kappa^\pm}(O_1 \cup O_2, \widehat{g}) + \mathcal{R}^{\kappa^\pm}(O_1 \cap O_2, \widehat{g})$.

Let $\widetilde{O}_1, \widetilde{O}_2 \in \mathcal{A}(M)$ such that $\widetilde{O}_1 \subset \subset O_1$ and $\widetilde{O}_2 \subset \subset O_2$. Observe that $\widetilde{O}_1 \cup \widetilde{O}_2$ and $\widetilde{O}_1 \cap \widetilde{O}_2$ are compactly contained in $O_1 \cup O_2$ and $O_1 \cap O_2$ respectively. Then there exist $\widetilde{O}_3, \widetilde{O}_4 \in \mathcal{A}(M)$ such that $\widetilde{O}_1 \cup \widetilde{O}_2 \subset \subset \widetilde{O}_3 \subset \subset O_1 \cup O_2$ and $\widetilde{O}_1 \cap \widetilde{O}_2 \subset \subset \widetilde{O}_4 \subset \subset O_1 \cap O_2$. Thus

$$\begin{aligned} \mathcal{R}^{\kappa^\pm}(\widetilde{O}_1, \widehat{g}) + \mathcal{R}^{\kappa^\pm}(\widetilde{O}_2, \widehat{g}) &\leq \mathcal{R}^{\kappa^\pm}(\widetilde{O}_3, \widehat{g}) + \mathcal{R}^{\kappa^\pm}(\widetilde{O}_4, \widehat{g}) \\ &\leq \mathcal{R}^{\kappa^\pm}(O_1 \cup O_2, \widehat{g}) + \mathcal{R}^{\kappa^\pm}(O_1 \cap O_2, \widehat{g}) \end{aligned}$$

due to item (6) of Definition 8.1.

Claim 2: $\mathcal{R}^{\kappa^\pm}(O_1 \cup O_2, \widehat{g}) + \mathcal{R}^{\kappa^\pm}(O_1 \cap O_2, \widehat{g}) \leq \mathcal{R}^{\kappa^\pm}(O_1, \widehat{g}) + \mathcal{R}^{\kappa^\pm}(O_2, \widehat{g})$.

It follows in a similar fashion as Claim 1. Let $\tilde{O}_3, \tilde{O}_4 \in \mathcal{A}(M)$ such that $\tilde{O}_3 \subset\subset O_1 \cup O_2$ and $\tilde{O}_4 \subset\subset O_1 \cap O_2$. Then we can find $\tilde{O}_1 \subset O_1$ and $\tilde{O}_2 \subset O_2$ in $\mathcal{A}(M)$ such that $\tilde{O}_3 \subset\subset \tilde{O}_1 \cup \tilde{O}_2$ and $\tilde{O}_4 \subset\subset \tilde{O}_1 \cap \tilde{O}_2$. Thus

$$\begin{aligned} \mathcal{R}^{\kappa^\pm}(\tilde{O}_3, \hat{g}) + \mathcal{R}^{\kappa^\pm}(\tilde{O}_4, \hat{g}) &\leq \mathcal{R}^{\kappa^\pm}(\tilde{O}_1, \hat{g}) + \mathcal{R}^{\kappa^\pm}(\tilde{O}_2, \hat{g}) \\ &\leq \mathcal{R}^{\kappa^\pm}(O_1, \hat{g}) + \mathcal{R}^{\kappa^\pm}(O_2, \hat{g}). \end{aligned}$$

□

LEMMA 8.6. *Let $\{O_i\}_{i \in \mathbb{N}}$ be a decreasing sequence of open sets such that $\cap_{i=1}^\infty O_i = \emptyset$. Then*

$$\lim_{i \rightarrow \infty} \mathcal{R}^{\kappa^\pm}(O_i, \hat{g}) = 0.$$

Proof. For every $i \in \mathbb{N}$, there exist a $\tilde{O}_i \in \mathcal{A}(M)$ such that $\tilde{O}_i \subset\subset O_i$ and $\mathcal{R}^{\kappa^\pm}(\tilde{O}_i, \hat{g}) > \mathcal{R}^{\kappa^\pm}(O_i, \hat{g}) - 1/i$. If we write $V_i = \cup_{j=i}^\infty \tilde{O}_j$, then $\cap_{i=1}^\infty V_i = \emptyset$. Thus

$$\lim_{i \rightarrow \infty} \mathcal{R}^{\kappa^\pm}(O_i, \hat{g}) \leq \lim_{i \rightarrow \infty} \left(\mathcal{R}^{\kappa^\pm}(\tilde{O}_i, \hat{g}) + \frac{1}{i} \right) = 0$$

due to Property (5) of Definition 8.1.

□

We extend $\mathcal{R}^{\kappa^\pm}(\cdot, \hat{g})$ to 2^M defining

$$(34) \quad \mathcal{R}^{\kappa^\pm}(A, \hat{g}) = \inf_{O \in \mathcal{T}(M), O \supset A} \{ \mathcal{R}^{\kappa^\pm}(O, \hat{g}) \},$$

and

$$(35) \quad \mathcal{R}^\kappa(A, \hat{g}) = \mathcal{R}^{\kappa^+}(A, \hat{g}) - \mathcal{R}^{\kappa^-}(A, \hat{g}).$$

Notice that Definition (35) coincides with Definition (33) if $A \in \mathcal{T}(M)$. It is well known from the classical measure theory that (34) does not define a measure on 2^M because it fails to be countably additive.

Define

$$(36) \quad \tilde{\mathcal{B}}(M) := \{ A \in 2^M; \mathcal{R}^{\kappa^\pm}(A, \hat{g}) = \sup_{C \subset A; C \text{ compact}} \mathcal{R}^{\kappa^\pm}(C, \hat{g}) \}.$$

We will prove that $\tilde{\mathcal{B}}(M)$ is a σ -algebra that contains $\mathcal{B}(M)$ and that the triple $(M, \tilde{\mathcal{B}}(M), \mathcal{R}^\kappa(\cdot, \hat{g}))$ is a signed measure space. The full proof of this assertion is quite long, although not difficult. It uses just elementary classical measure theory and it will be settled in Theorem 8.10.

LEMMA 8.7. *Let (M, \hat{g}) be a closed non-regular Riemannian manifold with a curvature measure generator $(\mathcal{A}(M), \mathcal{R}^{\kappa^+}(\cdot, \hat{g}), \mathcal{R}^{\kappa^-}(\cdot, \hat{g}))$. Then $\tilde{\mathcal{B}}(M)$ contains every open subset of M and every closed (compact) subset of M .*

Proof. Let $O \subset M$ be an open set. Then there exist an increasing sequence of compact sets $\{C_i\}_{i=1}^\infty$ such that $\cup_{i=1}^\infty C_i = O$. Fix $\epsilon > 0$. Observe that $\{O - C_i\}_{i=1}^\infty$ is a sequence of open sets such that their intersection vanishes. Then $\lim_{i \rightarrow \infty} \mathcal{R}^{\kappa^\pm}(O - C_i, \hat{g}) = 0$ due to Lemma 8.6. This implies that there exists an $N \in \mathbb{N}$ such that $\mathcal{R}^{\kappa^\pm}(O - C_i, \hat{g}) < \epsilon/2$ for every $i \geq N$. We also have that there exists an open set \tilde{O} such that $O \supset\supset \tilde{O} \supset C_N$ and $\mathcal{R}^{\kappa^\pm}(\tilde{O}, \hat{g}) < \mathcal{R}^{\kappa^\pm}(C_N, \hat{g}) + \epsilon/2$. But Lemma 8.5 implies that $\mathcal{R}^{\kappa^\pm}(\tilde{O}, \hat{g}) + \mathcal{R}^{\kappa^\pm}(O -$

$C_N, \hat{g}) = \mathcal{R}^{\kappa^\pm}(O, \hat{g}) + \mathcal{R}^{\kappa^\pm}(\tilde{O} \cap (O - C_N), \hat{g}) > \mathcal{R}^{\kappa^\pm}(O, \hat{g})$. Consequently we have that $\mathcal{R}^{\kappa^\pm}(O, \hat{g}) < \mathcal{R}^{\kappa^\pm}(C_N, \hat{g}) + \epsilon$ and $\tilde{\mathcal{B}}(M)$ contains every open set of M .

Finally notice that every compact subset of M lies in $\tilde{\mathcal{B}}(M)$ by definition, what settles the lemma. \square

LEMMA 8.8. *Let (M, \hat{g}) be a closed non-regular Riemannian manifold with a Lipschitz-Killing curvature measure generator $(\mathcal{A}(M), \mathcal{R}^{\kappa^+}(\cdot, \hat{g}), \mathcal{R}^{\kappa^-}(\cdot, \hat{g}))$ and let $O \subset M$ be an open set. Then $\mathcal{R}^{\kappa^\pm}(M, \hat{g}) = \mathcal{R}^{\kappa^\pm}(O, \hat{g}) + \mathcal{R}^{\kappa^\pm}(M - O, \hat{g})$.*

Proof. Let $\tilde{O} \subset M$ be an open set such that $\tilde{O} \supset M - O$. Then

$$\mathcal{R}^{\kappa^\pm}(M, \hat{g}) = \mathcal{R}^{\kappa^\pm}(O, \hat{g}) + \mathcal{R}^{\kappa^\pm}(\tilde{O}, \hat{g}) - \mathcal{R}^{\kappa^\pm}(O \cap \tilde{O}, \hat{g}),$$

due to Lemma 8.5. If we take a decreasing sequence of open sets $\{\tilde{O}_i\}_{i=1, \dots, \infty}$ such that $\bigcap_{i=1}^{\infty} \tilde{O}_i = M - O$, then

$$\begin{aligned} \mathcal{R}^{\kappa^\pm}(M, \hat{g}) &= \lim_{i \rightarrow \infty} \left[\mathcal{R}^{\kappa^\pm}(O, \hat{g}) + \mathcal{R}^{\kappa^\pm}(\tilde{O}_i, \hat{g}) - \mathcal{R}^{\kappa^\pm}(O \cap \tilde{O}_i, \hat{g}) \right] \\ &= \mathcal{R}^{\kappa^\pm}(O, \hat{g}) + \mathcal{R}^{\kappa^\pm}(M - O, \hat{g}) \end{aligned}$$

due to Lemma 8.6. \square

LEMMA 8.9. *Let (M, \hat{g}) be a closed non-regular Riemannian manifold with Lipschitz-Killing curvature measure generator $(\mathcal{A}(M), \mathcal{R}^{\kappa^+}(\cdot, \hat{g}), \mathcal{R}^{\kappa^-}(\cdot, \hat{g}))$. Let $\{A_i\}_{i=1, \dots, \infty}$ be a countable collection of subsets of M . Denote $A = \bigcup_{i=1}^{\infty} A_i$. Then*

- (1) $\mathcal{R}^{\kappa^\pm}(A, \hat{g}) \leq \sum_{i=1}^{\infty} \mathcal{R}^{\kappa^\pm}(A_i, \hat{g})$;
- (2) *If $\{A_i\}_{i=1, \dots, N}$ is a finite collection of pairwise disjoint closed sets, then $\mathcal{R}^{\kappa^\pm}(A, \hat{g}) = \sum_{i=1}^N \mathcal{R}^{\kappa^\pm}(A_i, \hat{g})$.*

Proof.

Assertion (1)

Fix $\epsilon > 0$. For each A_i there exists an open set $O_i \supset A_i$ such that $\mathcal{R}^{\kappa^\pm}(O_i, \hat{g}) < \mathcal{R}^{\kappa^\pm}(A_i, \hat{g}) + \epsilon \cdot 2^{-i}$. Denote $O = \bigcup_{i=1}^{\infty} O_i$. From Lemma 8.5, it follows that $\mathcal{R}^{\kappa^\pm}(O, \hat{g}) \leq \sum_{i=1}^{\infty} \mathcal{R}^{\kappa^\pm}(O_i, \hat{g})$. Therefore O is an open set that contains A and satisfies $\mathcal{R}^{\kappa^\pm}(O, \hat{g}) < \sum_{i=1}^{\infty} \mathcal{R}^{\kappa^\pm}(A_i, \hat{g}) + \epsilon$, what proves this assertion.

Assertion (2)

Fix $\epsilon > 0$. Observe that we can choose pairwise disjoint open sets $\tilde{O}_i \supset A_i$ such that $\mathcal{R}^{\kappa^\pm}(\tilde{O}_i, \hat{g}) < \mathcal{R}^{\kappa^\pm}(A_i, \hat{g}) + \epsilon \cdot 2^{-i}$. Choose also an open set $\tilde{O} \supset A$ such that $\mathcal{R}^{\kappa^\pm}(\tilde{O}, \hat{g}) < \mathcal{R}^{\kappa^\pm}(A, \hat{g}) + \epsilon$. Now denote $O_i = \tilde{O}_i \cap \tilde{O}$ and $O = \bigcup_{i=1}^N O_i$. From Lemma 8.5, it follows

that $\mathcal{R}^{\kappa\pm}(O, \widehat{g}) = \sum_{i=1}^N \mathcal{R}^{\kappa\pm}(O_i, \widehat{g})$. Then

$$\begin{aligned} \sum_{i=1}^N \mathcal{R}^{\kappa\pm}(A_i, \widehat{g}) - \epsilon &\leq \sum_{i=1}^N \mathcal{R}^{\kappa\pm}(O_i, \widehat{g}) - \epsilon = \mathcal{R}^{\kappa\pm}(O, \widehat{g}) - \epsilon \\ &< \mathcal{R}^{\kappa\pm}(A, \widehat{g}) \leq \mathcal{R}^{\kappa\pm}(O, \widehat{g}) \\ &= \sum_{i=1}^N \mathcal{R}^{\kappa\pm}(O_i, \widehat{g}) \leq \sum_{i=1}^N \mathcal{R}^{\kappa\pm}(A_i, \widehat{g}) + \epsilon \end{aligned}$$

what proves this assertion. □

Finally we are able to prove the main theorem of this section:

THEOREM 8.10. *Let (M, \widehat{g}) be a closed non-regular Riemannian manifold. Let $(\mathcal{A}(M), \mathcal{R}^{\kappa+}(\cdot, \widehat{g}), \mathcal{R}^{\kappa-}(\cdot, \widehat{g}))$ be a Lipschitz-Killing curvature measure generator and consider $\widetilde{\mathcal{B}}(M)$ as defined in (36). Then*

- (1) $\widetilde{\mathcal{B}}(M)$ is a σ -algebra that contains $\mathcal{B}(M)$.
- (2) $(M, \widetilde{\mathcal{B}}(M), \mathcal{R}^{\kappa}(\cdot, \widehat{g}))$ is a (signed) measure space.

Proof. First we prove that $\widetilde{\mathcal{B}}(M)$ is a σ -algebra.

It is obvious that $M \in \widetilde{\mathcal{B}}(M)$.

Claim 1: If $A \in \widetilde{\mathcal{B}}(M)$, then $M - A \in \widetilde{\mathcal{B}}(M)$.

Suppose that $A \in \widetilde{\mathcal{B}}(M)$. Lemma 8.8 implies that

$$\begin{aligned} \mathcal{R}^{\kappa\pm}(M - A, \widehat{g}) &= \inf_{O \in \mathcal{T}(M), O \supset M-A} \mathcal{R}^{\kappa\pm}(O, \widehat{g}) \\ &= \inf_{O \in \mathcal{T}(M), O \supset M-A} [\mathcal{R}^{\kappa\pm}(M, \widehat{g}) - \mathcal{R}^{\kappa\pm}(M - O, \widehat{g})] \\ &= \mathcal{R}^{\kappa\pm}(M, \widehat{g}) - \sup_{O \in \mathcal{T}(M), O \supset M-A} \mathcal{R}^{\kappa\pm}(M - O, \widehat{g}) \\ &= \mathcal{R}^{\kappa\pm}(M, \widehat{g}) - \sup_{\substack{M-O \text{ compact}, \\ M-O \subset A}} \mathcal{R}^{\kappa\pm}(M - O, \widehat{g}) \\ &= \mathcal{R}^{\kappa\pm}(M, \widehat{g}) - \mathcal{R}^{\kappa\pm}(A, \widehat{g}) \\ &= \mathcal{R}^{\kappa\pm}(M, \widehat{g}) - \inf_{O \in \mathcal{T}(M), O \supset A} \mathcal{R}^{\kappa\pm}(O, \widehat{g}) \\ &= \mathcal{R}^{\kappa\pm}(M, \widehat{g}) - \inf_{\substack{M-O \text{ compact}, \\ M-O \subset M-A}} \mathcal{R}^{\kappa\pm}(O, \widehat{g}) \\ &= \sup_{\substack{M-O \text{ compact}, \\ M-O \subset M-A}} \mathcal{R}^{\kappa\pm}(M, \widehat{g}) - \mathcal{R}^{\kappa\pm}(O, \widehat{g}) \\ &= \sup_{\substack{M-O \text{ compact}, \\ M-O \subset M-A}} \mathcal{R}^{\kappa\pm}(M - O, \widehat{g}). \end{aligned}$$

Thus $M - A \in \widetilde{\mathcal{B}}(M)$.

Claim 2: If $A_1, A_2 \in \widetilde{\mathcal{B}}(M)$, then $A_1 \cap A_2 \in \widetilde{\mathcal{B}}(M)$.

For every $\epsilon > 0$, there exist an open set $O_i \supset A_i$ and a compact set $C_i \subset A_i$, $i = 1, 2$, such that

$$\mathcal{R}^{\kappa\pm}(O_i, \widehat{g}) - \frac{\epsilon}{4} < \mathcal{R}^{\kappa\pm}(A_i, \widehat{g}) < \mathcal{R}^{\kappa\pm}(C_i, \widehat{g}) + \frac{\epsilon}{4},$$

what gives

$$\mathcal{R}^{\kappa\pm}(O_1, \widehat{g}) + \mathcal{R}^{\kappa\pm}(O_2, \widehat{g}) - \frac{\epsilon}{2} < \mathcal{R}^{\kappa\pm}(C_1, \widehat{g}) + \mathcal{R}^{\kappa\pm}(C_2, \widehat{g}) + \frac{\epsilon}{2}.$$

Lemma 8.5 implies

$$\mathcal{R}^{\kappa\pm}(O_1 \cup O_2, \widehat{g}) + \mathcal{R}^{\kappa\pm}(O_1 \cap O_2, \widehat{g}) - \frac{\epsilon}{2} = \mathcal{R}^{\kappa\pm}(O_1, \widehat{g}) + \mathcal{R}^{\kappa\pm}(O_2, \widehat{g}) - \frac{\epsilon}{2},$$

and using Lemma 8.8 we have that

$$\begin{aligned} & \mathcal{R}^{\kappa\pm}(O_1 \cap O_2, \widehat{g}) - \frac{\epsilon}{2} \\ & < \mathcal{R}^{\kappa\pm}(C_1, \widehat{g}) + \mathcal{R}^{\kappa\pm}(C_2, \widehat{g}) - \mathcal{R}^{\kappa\pm}(O_1 \cup O_2, \widehat{g}) + \frac{\epsilon}{2} \\ & = \mathcal{R}^{\kappa\pm}(M, \widehat{g}) - \mathcal{R}^{\kappa\pm}(M - C_1, \widehat{g}) + \mathcal{R}^{\kappa\pm}(M, \widehat{g}) - \mathcal{R}^{\kappa\pm}(M - C_2, \widehat{g}) \\ & \quad - \mathcal{R}^{\kappa\pm}(O_1 \cup O_2, \widehat{g}) + \frac{\epsilon}{2} \\ & = 2\mathcal{R}^{\kappa\pm}(M, \widehat{g}) - \mathcal{R}^{\kappa\pm}(M - (C_1 \cap C_2), \widehat{g}) - \mathcal{R}^{\kappa\pm}(M - (C_1 \cup C_2), \widehat{g}) \\ & \quad - \mathcal{R}^{\kappa\pm}(O_1 \cup O_2, \widehat{g}) + \frac{\epsilon}{2} \\ & = \mathcal{R}^{\kappa\pm}(C_1 \cap C_2, \widehat{g}) + \mathcal{R}^{\kappa\pm}(C_1 \cup C_2, \widehat{g}) - \mathcal{R}^{\kappa\pm}(O_1 \cup O_2, \widehat{g}) + \frac{\epsilon}{2} \\ & < \mathcal{R}^{\kappa\pm}(C_1 \cap C_2, \widehat{g}) + \frac{\epsilon}{2}. \end{aligned}$$

Now notice that $(C_1 \cap C_2) \subset (A_1 \cap A_2)$ and

$$\mathcal{R}^{\kappa\pm}(A_1 \cap A_2, \widehat{g}) < \mathcal{R}^{\kappa\pm}(O_1 \cap O_2, \widehat{g}) < \mathcal{R}^{\kappa\pm}(C_1 \cap C_2, \widehat{g}) + \epsilon.$$

Thus $A_1 \cap A_2 \in \widetilde{\mathcal{B}}(M)$.

Claim 3: If $\{A_i\}_{i=1\dots\infty}$ is a *disjoint* countable collection of subsets in $\widetilde{\mathcal{B}}(M)$, then $A := \bigcup_{i=1}^{\infty} A_i \in \widetilde{\mathcal{B}}(M)$.

Fix $\epsilon > 0$. Then for every i there exist a compact set $C_i \subset A_i$ such that

$$\mathcal{R}^{\kappa\pm}(A_i, \widehat{g}) < \mathcal{R}^{\kappa\pm}(C_i, \widehat{g}) + \epsilon \cdot 2^{-i-1}$$

what implies

$$(37) \quad \sum_{i=1}^{\infty} \mathcal{R}^{\kappa\pm}(A_i, \widehat{g}) < \sum_{i=1}^{\infty} \mathcal{R}^{\kappa\pm}(C_i, \widehat{g}) + \epsilon/2.$$

Observe that there exist $N \in \mathbb{N}$ such that

$$(38) \quad \sum_{i=1}^{\infty} \mathcal{R}^{\kappa\pm}(C_i, \widehat{g}) < \sum_{i=1}^N \mathcal{R}^{\kappa\pm}(C_i, \widehat{g}) + \epsilon/2.$$

Denote $C = \bigcup_{i=1}^N C_i$. Eqs. (37) and (38) and Lemma 8.9 gives

$$(39) \quad \begin{aligned} \mathcal{R}^{\kappa\pm}(A, \widehat{g}) & \leq \sum_{i=1}^{\infty} \mathcal{R}^{\kappa\pm}(A_i, \widehat{g}) & \leq \sum_{i=1}^{\infty} \mathcal{R}^{\kappa\pm}(C_i, \widehat{g}) + \epsilon/2 \\ & \leq \sum_{i=1}^N \mathcal{R}^{\kappa\pm}(C_i, \widehat{g}) + \epsilon & = \mathcal{R}^{\kappa\pm}(C, \widehat{g}) + \epsilon. \end{aligned}$$

We found a compact set $C \subset A$ such that the inequality above is satisfied, what implies that $A \in \widetilde{\mathcal{B}}(M)$.

Claim 4: If $\{A_i\}_{i=1\dots\infty}$ is *any* countable collection of subsets in $\widetilde{\mathcal{B}}(M)$, then $\bigcup_{i=1}^{\infty} A_i \in \widetilde{\mathcal{B}}(M)$.

Let $A_1, A_2 \in \tilde{\mathcal{B}}(M)$. Using Claims 1, 2 and 3, we can prove that $(A_1 - A_2)$ and $(A_2 - A_1)$ lie in $\tilde{\mathcal{B}}(M)$. Observe that $A_1 \cup A_2$ is the disjoint union $(A_1 \cap A_2) \cup (A_1 - A_2) \cup (A_2 - A_1)$ and it also lie in $\tilde{\mathcal{B}}(M)$. Similarly we can prove that $\bigcup_{i=1}^{\infty} A_i$ can be written as the union of disjoint sets in $\tilde{\mathcal{B}}(M)$. Consequently Claim 3 implies that $\bigcup_{i=1}^{\infty} A_i \in \tilde{\mathcal{B}}(M)$. Thus $\tilde{\mathcal{B}}(M)$ is a σ -algebra that contains $\mathcal{T}(M)$, what implies that it also contains $\mathcal{B}(M)$.

Claim 5: $(M, \tilde{\mathcal{B}}(M), \mathcal{R}^\kappa(\cdot, \hat{g}))$ is a (signed) measure space.

Take a pairwise disjoint sequence $\{A_i\}_{i=1 \dots \infty}$, $A_i \in \tilde{\mathcal{B}}(M)$. Eq. (39) gives

$$\mathcal{R}^{\kappa^\pm}(A, \hat{g}) \leq \sum_{i=1}^{\infty} \mathcal{R}^{\kappa^\pm}(A_i, \hat{g}) \leq \mathcal{R}^{\kappa^\pm}(C, \hat{g}) + \epsilon \leq \mathcal{R}^{\kappa^\pm}(A, \hat{g}) + \epsilon$$

and the equality

$$\mathcal{R}^{\kappa^\pm}(A, \hat{g}) = \sum_{i=1}^{\infty} \mathcal{R}^{\kappa^\pm}(A_i, \hat{g})$$

follows. □

DEFINITION 8.11. Let (M, \hat{g}) be a closed non-regular Riemannian manifold. Let $(\mathcal{A}(M), \mathcal{R}^{\kappa^+}(\cdot, \hat{g}), \mathcal{R}^{\kappa^-}(\cdot, \hat{g}))$ be a Lipschitz-Killing curvature measure generator and consider $\tilde{\mathcal{B}}(M)$ as defined in (36). The function $\mathcal{R}^\kappa(\cdot, \hat{g}) : \tilde{\mathcal{B}}(M) \rightarrow \mathbb{R}$ is the *Lipschitz-Killing curvature measure* of (M, \hat{g}) (with respect to $(\mathcal{A}(M), \mathcal{R}^{\kappa^+}(\cdot, \hat{g}), \mathcal{R}^{\kappa^-}(\cdot, \hat{g}))$).

REMARK 8.12. We defined the κ -th Lipschitz-Killing curvature measure from a collection $(\mathcal{A}(M), \mathcal{R}^{\kappa^+}(\cdot, \hat{g}), \mathcal{R}^{\kappa^-}(\cdot, \hat{g}))$ because it is useful in some situations. For instance, it is adequate to study the Gaussian curvature measure for piecewise smooth two-dimensional Riemannian manifolds (See Section 9). However we can consider other approaches if we want to study other problems. The most important thing here is to show that the mollifier smoothing with respect to \mathcal{P} is a useful tool to study non-regular Riemannian manifolds.

9. THE GAUSSIAN CURVATURE MEASURE FOR PIECEWISE SMOOTH TWO-DIMENSIONAL RIEMANNIAN MANIFOLDS

In this section we prove the existence of a “natural” Gaussian curvature measure generator for closed oriented piecewise smooth two-dimensional Riemannian manifolds. We get a Gaussian curvature measure $\mathcal{K}(\cdot, \hat{g}) : \mathcal{B}(M) \rightarrow \mathbb{R}$ for this kind of surfaces and we show that it has the expected values for some subsets of M (See Theorem 9.3). As a direct consequence, we get an alternative proof of the (well known) generalization of the Gauss-Bonnet theorem for this class of surfaces.

Let (M^2, \hat{g}) be a closed oriented piecewise smooth Riemannian manifold. Let $\Theta : \Upsilon \rightarrow M$ be a triangulation associated to (M, \hat{g}) . Fix a vertex $x \in V$. The point $\Theta^{-1}(x)$ lies to some simplexes, let us say Ξ_1, \dots, Ξ_k . For every triangle $\Theta(\Xi_i) \subset M$, we can associate the internal angle $\varsigma_i(x)$ of x . We suppose that $\varsigma_i(x) \neq 0$ for every $i = 1, \dots, k$. Consider the number $K_0(x) = 2\pi - \sum_{i=1}^k \varsigma_i(x)$. This is intuitively the “zero-dimensional curvature” of a vertex as explained in the example of the cube in the introduction.

Now consider a point $x \in E$. The point $\Theta^{-1}(x)$ lies to two simplexes Ξ_1 and Ξ_2 . The sum of the geodesic curvature of E at x with respect to $\Theta(\Xi_1)$ and $\Theta(\Xi_2)$ will be denoted by $K_1(x)$. This is intuitively the “one-dimensional curvature” of a point that lies on a edge of M .

Let $x \in F$. We denote the Gaussian curvature at x by $K_2(x)$.

Consider an open set $O \subset M$ such that its boundary ∂O is the image of a piecewise regular curve. Let $\alpha : [a, b] \rightarrow M$ be a positive regular parametrization of ∂O such that $\alpha(a) = \alpha(b)$. The external angle of ∂O at $\alpha(t_0) \in \partial O - V - E$ is the angle in the interval $(-\pi, \pi)$ formed by $\lim_{t \rightarrow t_0^-} \alpha'(t)$ and $\lim_{t \rightarrow t_0^+} \alpha'(t)$ (in this order). We will not allow the angles $\pm\pi$.

Although the following theorem is well known, we prove it here for the sake of completeness.

THEOREM 9.1. *Let (M^2, \hat{g}) be an closed and oriented piecewise smooth Riemannian manifold. Let $O \subset M$ be an open set with a piecewise smooth boundary ∂O such that:*

- (1) ∂O is smooth outside $\{q_1, \dots, q_k\}$.
- (2) ∂O does not intercept V .
- (3) $\{q_1, \dots, q_k\}$ does not intercept E .
- (4) ∂O intercepts E transversally (this includes the case $\partial O \cap E = \emptyset$).

Suppose that the external angles at q_i , $i=1, \dots, k$, are given by $\varrho(q_i)$. Then

$$(40) \quad \sum_{i=1}^k \varrho(q_i) + \int_{\partial O} k_{\hat{g}}(x).ds_{\hat{g}} \\ = 2\pi \cdot \chi(O \cup \partial O) - \sum_{O \cap V} K_0(x) - \int_{O \cap E} K_1(x).ds_{\hat{g}} - \int \int_{O \cap F} K_2(x).dV_{\hat{g}}$$

where $k_{\hat{g}}$ is the geodesic curvature of ∂O with respect to O , $ds_{\hat{g}}$ denotes the length element and $\chi(O \cup \partial O)$ denotes the Euler characteristic of $O \cup \partial O$.

Proof. This proof is analogous to the proof of the classical Gauss-Bonnet Theorem (For instance, see [20]).

First of all, we can take a triangulation $\Theta' : \Upsilon' \rightarrow (O \cup \partial O)$ associated to $(O \cup \partial O, \hat{g}|_{O \cup \partial O})$. Let $\#F$, $\#E$ and $\#V$ be the number of faces, edges and vertices of Υ' . Let $\#B$ be the number of vertices (and edges) on the boundary of Υ' . Formula (40) holds for a simplex Ξ' , that is,

$$\int \int_{\Theta'(\Xi')} K_2(x).dV_{\hat{g}} = -\pi - \int_{\Theta'(\partial \Xi')} k_{\hat{g}}(x).ds_{\hat{g}} + \sum_{i=1}^3 \varsigma_i.$$

Summing up the formula above for every simplex, we have that

$$(41) \quad \int \int_O K_2(x).dV_{\hat{g}} \\ = -3\pi(\#F) + 2\pi(\#F) - \int_{O \cap E} K_1(x).ds_{\hat{g}} - \int_{\partial O} k_{\hat{g}}(x).ds_{\hat{g}} \\ + \sum_{V \in O} \sum \varsigma_i + \sum_{V \in \partial O} \sum \varsigma_i = (*).$$

But $3(\#F) = 2(\#E) - (\#B)$, what implies that

$$\begin{aligned}
(*) &= -2\pi(\#E) + 2\pi(\#F) + 2\pi(\#V) - \int_{O \cap E} K_1(x).ds_{\hat{g}} - \int_{\partial O} k_g(x).ds_{\hat{g}} \\
&\quad - \sum_{V \in O} \left(2\pi - \sum s_i \right) - \sum_{V \in \partial O} \left(\pi - \sum s_i \right) \\
&= 2\pi \cdot \chi(O \cup \partial O) - \int_{O \cap E} K_1(x).ds_{\hat{g}} - \int_{\partial O} k_g(x).ds_{\hat{g}} \\
&\quad - \sum_{V \in O} K_0(x) - \sum_{V \in \partial O} \varrho(q_i)
\end{aligned}$$

what settles the Theorem. □

The following generalization of the classical Gauss-Bonnet Theorem is an immediate consequence of the proof of Theorem 9.1.

THEOREM 9.2. *Let (M^2, \hat{g}) be a closed and oriented piecewise smooth Riemannian manifold. Then*

$$2\pi \cdot \chi(M) = \sum_V K_0(x) + \int_E K_1(x).ds_{\hat{g}} + \int \int_F K_2(x).dV_{\hat{g}}$$

where $ds_{\hat{g}}$ is the length element of the edge.

The next theorem states that for piecewise smooth two-dimensional Riemannian manifold, $K_0(x)$, $K_1(x)$ and $K_2(x)$ can be really interpreted as the zero dimensional curvature of the vertex, the one dimensional curvature of the edge and the two dimensional curvature of the face respectively (See Eq. (43)).

THEOREM 9.3. *Let (M^2, \hat{g}) be an closed and oriented piecewise smooth Riemannian manifold. Let $O \subset M$ be an open set with a piecewise smooth boundary ∂O such that:*

- (1) ∂O is smooth outside $\{q_1, \dots, q_k\}$.
- (2) ∂O does not intercept V .
- (3) $\{q_1, \dots, q_k\}$ does not intercept E .
- (4) ∂O intercepts E transversally (this includes the case $\partial O \cap E = \emptyset$).

Let \mathcal{P} be a locally finite covering $\{(U_\omega \subset\subset O_\omega, \tilde{e}_\omega)\}_{\omega \in \Lambda}$ together with the partition of unity $\{\psi_\omega\}_{\omega \in \Lambda}$ subordinated to $\{U_\omega\}_{\omega \in \Lambda}$. Let $\hat{g}_{\varepsilon, \mathcal{P}}$ be the mollifier smoothing of \hat{g} with respect to \mathcal{P} . Suppose that the external angles of ∂O at $q_i \in (M, \hat{g}_{\varepsilon, \mathcal{P}})$, $i=1, \dots, k$, is given by $\varrho_\varepsilon(q_i)$. Then

$$(42) \quad \lim_{\varepsilon \rightarrow 0} \left(\sum_{i=1}^k \varrho_\varepsilon(q_i) + \int_{\partial O} k_{g_{\varepsilon, \mathcal{P}}}(x).ds_{\hat{g}} \right) = \sum_{i=1}^k \varrho(q_i) + \int_{\partial O} k_{\hat{g}}(x).ds_{\hat{g}}$$

and

$$(43) \quad \lim_{\varepsilon \rightarrow 0} \left(\int \int_O K_2(x).dV_{\hat{g}_{\varepsilon, \mathcal{P}}} \right) = \sum_{O \cap V} K_0(x) + \int_{O \cap E} K_1(x).ds_{\hat{g}} + \int \int_{O \cap F} K_2(x).dV_{\hat{g}}.$$

Proof. We have that

$$\lim_{\varepsilon \rightarrow 0} \left(\sum_{i=1}^k \varrho_\varepsilon(q_i) \right) = \sum_{i=1}^k \varrho(q_i)$$

because $\widehat{g}_{\varepsilon, \mathcal{P}}$ converges to \widehat{g} on F .

In order to see that

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial O} k_{\widehat{g}_{\varepsilon, \mathcal{P}}}(x).ds_{\widehat{g}} = \int_{\partial O} k_{\widehat{g}}(x).ds_{\widehat{g}}$$

we split the integral through ∂O in two parts: the integral near the edges and the integral far from the edges.

Let $y \in E \cap \partial O$. Theorem 7.8 implies that

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial O \cap V_y} k_{\widehat{g}_{\varepsilon, \mathcal{P}}}(x).ds_{\widehat{g}}$$

can be made as small as we want if we choose a sufficiently small neighborhood V_y of y .

Theorem 5.4 implies that the Levi-Civita connection of $\widehat{g}_{\varepsilon, \mathcal{P}}$ converges uniformly to the Levi-Civita connection of \widehat{g} on compact subsets of F . Then

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial O \cap \widetilde{O}} k_{\widehat{g}_{\varepsilon, \mathcal{P}}}(x).ds_{\widehat{g}} = \int_{\partial O \cap \widetilde{O}} k_{\widehat{g}}(x).ds_{\widehat{g}}$$

for every open set $\widetilde{O} \subset \subset F$.

Therefore

$$\lim_{\varepsilon \rightarrow 0} \int_{\partial O} k_{\widehat{g}_{\varepsilon, \mathcal{P}}}(x).ds_{\widehat{g}} = \int_{\partial O} k_{\widehat{g}}(x).ds_{\widehat{g}}.$$

and Eq. (42) follows. Equation (43) is a direct consequence of (42) and Theorem 9.1. \square

Let (M^2, \widehat{g}) be a closed and oriented piecewise smooth Riemannian manifold. Let us define a Gaussian curvature measure generator. Define $\mathcal{A}(M)$ as the family of open sets that satisfies the conditions given in Theorem 9.1. Define $K_0^-(x) = \max(-K_0(x), 0)$, $K_0^+(x) = \max(K_0(x), 0)$, $K_1^-(x) = \max(-K_1(x), 0)$, $K_1^+(x) = \max(K_1(x), 0)$, $K_2^-(x) = \max(-K_2(x), 0)$, $K_2^+(x) = \max(K_2(x), 0)$. Define $\mathcal{K}^\pm(\cdot, \widehat{g}) : \mathcal{A}(M) \rightarrow \mathbb{R}$ as

$$(44) \quad \mathcal{K}^\pm(\widetilde{O}, \widehat{g}) := \sum_{\widetilde{O} \cap V} K_0^\pm(x) + \int_{\widetilde{O} \cap E} K_1^\pm(x).ds_{\widehat{g}} + \int \int_{\widetilde{O} \cap F} K_2^\pm(x).dV_{\widehat{g}},$$

where the superscripts of K_i^\pm and \mathcal{K}^\pm are explained in Remark 8.2.

THEOREM 9.4. *The triple $(\mathcal{A}(M), \mathcal{K}^-(\cdot, \widehat{g}), \mathcal{K}^+(\cdot, \widehat{g}))$ defined by Eq. (44) is a Gaussian curvature measure generator.*

Proof. Properties (1), (3), (4), (5), (6) and (7) of Definition 8.1 hold due to the definition of $\mathcal{A}(M)$, (43) and (44).

Let us see that Property (2) holds: Let $O \in \mathcal{T}(M)$. Denote $C = M - O$. Put an arbitrary smooth Riemannian metric \check{g} on M . Define

$$C(n) = \{x \in O; d_{\check{g}}(x, C) \geq 1/n\}$$

for every $n \in \mathbb{N}$. Notice that $C(n)$ is a compact set. Cover $C(n)$ by elements of $\mathcal{A}(M)$ such that they are compactly contained on O . Take a finite subcover $\{O_1, \dots, O_k\}$ of this covering. Now notice that it is possible to make small deformations in O_i , $i = 1, \dots, k$, in such a way that

- (1) $C(n) \subset\subset \bigcup_{i=1}^k O_i \subset\subset O$.
- (2) The vertices of O_j do not intercept vertices and edges of other open sets O_l , $l \neq j$.
- (3) All the edges of the open sets O_i , $i = 1, \dots, k$, intercept themselves transversally.
- (4) All the edges of the open sets O_i , $i = 1, \dots, k$, intercept E transversally.
- (5) ∂O_i do not intercept V for every $i = 1, \dots, k$.

Observe that $\bigcup_{i=1}^k O_i$ is an open set satisfying all the conditions given in Theorem 9.1

except, eventually, (3). But we can make a further perturbation on $\tilde{O}(n) := \bigcup_{i=1}^k O_i$ in order to satisfy (3). Then, for every $n \in \mathbb{N}$, we can find an open set $\tilde{O}(n) \in \mathcal{A}(M)$ such that $C(n) \subset\subset \tilde{O}(n) \subset\subset O$ and these sets can be built in such a way that $\tilde{O}(n) \subset \tilde{O}(n+1)$. Therefore Property (2) of Definition 8.1 holds and the triple $(\mathcal{A}(M), \mathcal{K}^+(\cdot, \hat{g}), \mathcal{K}^-(\cdot, \hat{g}))$ is a Gaussian curvature measure generator. \square

Theorem 8.10 implies that the Gaussian curvature measure is extendable for every Borel set. If $O \in \mathcal{T}(M)$, then it is not difficult to see that

$$(45) \quad \mathcal{K}^\pm(O, \hat{g}) = \sum_{O \cap V} K_0^\pm(x) + \int_{O \cap E} K_1^\pm(x) \cdot ds_{\hat{g}} + \int \int_{O \cap F} K_2^\pm(x) \cdot dV_{\hat{g}}.$$

It is not difficult to see either that if $x \in V$, then

$$(46) \quad \mathcal{K}^\pm(\{x\}, \hat{g}) = K_0(x)$$

and if $v \subset E$, then

$$(47) \quad \mathcal{K}^\pm(v, \hat{g}) = \int_v K_1(x) \cdot ds_{\hat{g}}.$$

Therefore $\mathcal{K}(\cdot, \hat{g}) : \mathcal{B}(M) \rightarrow \mathbb{R}$ is a Gaussian curvature measure which gives the expected geometrical values for some subsets of M .

10. CURVATURE DIMENSION AND AN INSTRUCTIVE NON-REGULAR EXAMPLE

In Section 9, we studied piecewise smooth two-dimensional Riemannian manifolds. In particular, (46) and (47) show that the “curvature integra” can be calculated in subsets with dimension less than two. Moreover V , E and F are the “natural” subsets where the “zero-dimensional curvature”, “one-dimensional curvature” and the “two-dimensional curvature” arise respectively. A natural question here is whether the “curvature integra” can arise naturally in subsets with non-integer Hausdorff dimension (For Hausdorff dimension, see, for instance, [19]). In this Section, we present a natural candidate to answer this question affirmatively.

When we study spaces with non-integer Hausdorff dimension, the natural starting point is the classical Cantor ternary set, which we denote by \mathcal{C} . It is the subset of $[0, 1]$ which is composed by numbers that have some ternary representation without the number ‘1’ (See, for instance, [17]). It is well known that the Hausdorff dimension of \mathcal{C} is equal to $\ln 2 / \ln 3$ (See, for instance, [13]). One easy way to understand \mathcal{C} is to take out the open middle third interval (that is, $(1/3, 2/3)$) from $[0, 1]$. After that, we take out the open middle third intervals from every remaining connected set (At this point, the connected

sets we are talking about are $[0, 1/3]$ and $[2/3, 1]$). We do it recursively and the remaining set is \mathcal{C} .

We have the Cantor ternary function associated with \mathcal{C} , which we denote by $f_{\mathcal{C}} : [0, 1] \rightarrow \mathbb{R}$. Let $x = 0.a_1a_2a_3\dots$ be a ternary representation of $x \in [0, 1]$. Denote by $N : [0, 1] \rightarrow \mathbb{N}$ the first i such that $a_i = 1$. If the number '1' is absent from the ternary representation of x , make $N(x) = \infty$. Now define

$$f_{\mathcal{C}}(x) = \frac{1}{2} \sum_{i=1}^{N(x)} \frac{a_i}{2^i}.$$

It is not difficult to prove that $f_{\mathcal{C}}$ is well defined. Moreover it is a continuous increasing function such that its derivative is equal to zero outside \mathcal{C} .

We will built a curve of class C^1 in \mathbb{R}^2 which is homeomorphic to a circle using $f_{\mathcal{C}}$. It will be flat almost everywhere and the curvature will be concentrated in a set with non-integer dimension. Define $\theta_{\mathcal{C}} : [0, 1] \rightarrow \mathbb{R}$ by

$$\theta_{\mathcal{C}}(t) = \begin{cases} 2\pi f_{\mathcal{C}}(\frac{4t}{3}) & \text{if } t \in [0, \frac{3}{4}] \\ 2\pi & \text{if } t \in (\frac{3}{4}, 1] \end{cases}$$

and $\gamma_{\mathcal{C}} : [0, 1] \rightarrow \mathbb{R}^2$ by

$$\gamma_{\mathcal{C}}(s) = \left(\int_0^s \cos \theta_{\mathcal{C}}(t) dt, \int_0^s \sin \theta_{\mathcal{C}}(t) dt \right).$$

$\gamma_{\mathcal{C}}$ is a closed curve parametrized by arclength and the angle that its tangent vector does with the vector $(1, 0) \in \mathbb{R}^2$ is $\theta_{\mathcal{C}}$. Notice that it is flat outside \mathcal{C} , that is, its curvature is concentrated on \mathcal{C} . Now we compare the nature of the curvature of $\gamma_{\mathcal{C}}$ and the curvature of some well known examples.

Consider a piecewise regular closed simple planar curve $\alpha : [a, b] \rightarrow \mathbb{R}^2$. Denote the points where the curve fails to be smooth by $p_1, \dots, p_k \in [a, b]$. Suppose that α is oriented positively and that it is parametrized by arclength outside $p_1, \dots, p_k \in [a, b]$. At each p_i , $1 \leq i \leq k$, we can associate an angle $K_0(p_i)$ which is the angle between the vectors $\lim_{t \rightarrow p_i^-} \alpha'(t)$ and $\lim_{t \rightarrow p_i^+} \alpha'(t)$. If K_1 is the geodesic curvature of α , it is well known that

$$\sum_{i=1}^k K_0(p_i) + \int_{\alpha} K_1(t) dt = 2\pi,$$

where the integral is done with respect to the arclength of α .

K_0 and K_1 can be seen as the zero dimensional and one dimensional curvature respectively, that is, the curvature that is concentrated in zero dimensional and one dimensional subsets respectively. Alternatively, they can be defined as

$$K_d(p) = \lim_{t \rightarrow p^+, s \rightarrow p^-} \frac{\theta(t) - \theta(s)}{(t - s)^d},$$

where $\theta(t)$ is the angle that the tangent vector at $\alpha(t)$ makes with some fixed vector in \mathbb{R}^2 (If $p \in \{a, b\}$, then these expressions can be adapted in a obvious way). The curvature dimension of a curve at a point p can be defined as the supremum of d such that

$$\lim_{t \rightarrow p^+, s \rightarrow p^-} \frac{\theta(t) - \theta(s)}{(t - s)^d}$$

is bounded. Observe that this definition has some similarities with the classical definition of Hausdorff dimension. It is not difficult to prove that the curvature dimension of $\gamma_{\mathcal{C}}$ at $t \in \mathcal{C}$ is $(\ln 2 / \ln 3)$.

Now we present a surface which curvature is concentrated in a non-integer (Hausdorff) dimensional set. Translate the image of $\gamma_{\mathcal{C}}$ in such a way that its center of mass is located at the origin of plane xy . Denote such a curve by $S_{\mathcal{C}}^1$. Observe that this curve is symmetric with respect to both axes. Now put the plane xy in the space xyz . Rotate $S_{\mathcal{C}}^1$ around the y -axis. We have a surface $S_{\mathcal{C}}^2$ homeomorphic to a bidimensional sphere.

Consider $S_{\mathcal{C}}^2$ with the induced metric \widehat{g} of \mathbb{R}^3 . It is a C^1 metric. Therefore it is non-regular in the sense of this work. Observe that the points outside the orbit of $\gamma_{\mathcal{C}}(\mathcal{C})$ are flat, and that the Hausdorff dimension of the orbit of $\gamma_{\mathcal{C}}(\mathcal{C})$ is $1 + \ln 2 / \ln 3$. Hence $(S_{\mathcal{C}}^2, \widehat{g})$ should have its curvature “concentrated” in a $(1 + (\ln 2 / \ln 3))$ dimensional set.

This sphere indicates that the curvature of a C^1 Riemannian manifold can have a strange behavior. We intend to study this sphere and other non-regular Riemannian manifolds in future works.

ACKNOWLEDGMENTS

The author would like to thank Professor Armando Caputi. His comments and ideas were very valuable at several points of this work.

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